Curvature

Tom LaGatta

University of Arizona

April 23, 2008
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Curvature of a Plane Curve

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Let $n$ be a normal vector field along $\gamma$. 

$K$ is $1$/radius of the osculating (kissing) circle.

$K$ is positive when the curve turns into $n$, but negative when turning away.
If $M$ is a surface smoothly embedded in $\mathbb{R}^3$, at each point $x$ there are two principal curvatures $k_1$ and $k_2$. 
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To calculate these at a point \( x \):

- Let \( n \) be the normal vector to \( M \) and \( P \) be any plane in \( \mathbb{R}^3 \) containing \( n \).
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- Let $n$ be the normal vector to $M$ and $P$ be any plane in $\mathbb{R}^3$ containing $n$.
- Calculate the curvature $K_P$ of the curve $\gamma_P = P \cap M$.
- Define

$$k_1 = \min_P K_P \quad \text{and} \quad k_2 = \max_P K_P.$$
The product

\[ K = k_1 k_2 \]

is called the Gaussian curvature of \( M \).
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**Theorem (Gauss, Theorema Egregium)**

\( K \) does not depend on the embedding of \( M \) in \( \mathbb{R}^3 \).

Thus curvature is an intrinsic property of a surface!
Some examples:

- The Euclidean plane $\mathbb{R}^2$ has zero curvature.
- The sphere $S^2$ has positive curvature.
- The hyperbolic plane $H^2$ has negative curvature.
Curvature of a Surface

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How does curvature affect geodesics, the “straight lines” of $M$?

- Positive curvature makes geodesics come together.
- Negative curvature forces them apart.

Geodesics are locally length-minimizing, but not necessarily globally! (Think of the sphere)
Knowing something about a local quantity at every point can give us a global result.

**Theorem (Gauss-Bonnet)**

If \( M \) is a compact surface, then

\[
\int_M K \, dA = 2\pi \chi(M),
\]

where \( \chi(M) \) is the Euler characteristic of \( M \).

The left side of the equation is local, since curvature is a pointwise function, whereas the right side is a global, topological quantity.
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Here is a nice application of Gauss-Bonnet:

**Corollary**

*Every surface in $\mathbb{R}^3$ not homeomorphic to $S^2$ has both positive and negative curvature.*
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- Let $M$ be any compact, smooth surface in $\mathbb{R}^3$ not homeomorphic to $S^2$.
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- $M$ is compact, so it must touch $S$. At this point, $K$ is positive.
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- By the classification of surfaces, \( \chi(M) = 2 - 2\text{ genus}(M) \leq 0 \).
- By Gauss-Bonnet,

\[
\int_M K \, dA = 2\pi \chi(M) \leq 0,
\]

so \( K \) must be negative somewhere. \( \Box \)
If we assume a uniform bound on curvature, we can say a lot about the topology of $M$. 

**Theorem (Bonnet-Myers)**

If $K(x) \geq \delta > 0$ for all $x \in M$, then $M$ is compact.

This justifies our intuition that positive curvature forces geodesics to come together.
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What about negative curvature?
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**Theorem (Hadamard-Cartan)**

If $K(x) \leq 0$ for all $x \in M$, then there is exactly one geodesic between any two points $x$ and $y$. Why does this make sense? A negatively curved space is locally everywhere like our toy model. If two geodesics start at a point, then they never come back together.
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If $K(x) \leq 0$ for all $x \in M$, and $M$ is simply connected (and complete),

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These theorems are extremely powerful, but they completely fail in situations of mixed curvature. Here is an example.

Surface in $\mathbb{R}^3$

$$(x, y, f(x, y))$$

Curvature profile

$K(x, y)$
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Mixed Curvature Example: Two Hills

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Curvature profile $K(x, y)$
Mixed Curvature Example: Bubble

In this example, the bubble has constant positive curvature at the top, but large negative curvature at the neck.

Surface of Revolution in $\mathbb{R}^3$

$$(f(\rho) \cos \theta, f(\rho) \sin \theta, g(\rho))$$
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$(f(\rho) \cos \theta, f(\rho) \sin \theta, g(\rho))$
M. Berger.
*A Panoramic View of Riemannian Geometry.*

M. do Carmo.
*Differential Geometry of Curves and Surfaces.*

J. Jost.
*Riemannian geometry and geometric analysis.*

J. Lee.
*Riemannian Manifolds: An Introduction to Curvature.*

B. O’Neill.
*Geodesics on Two Hills.*
http://www.math.ucla.edu/~bon/.