

# A hybrid domain decomposition method based on one-level FETI and BDDC algorithms

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## Abstract

A three-level domain decomposition is considered. Bodies in contact with each other are divided into subdomains, which in turn are the union of elements. Using an approach based purely on FETI (finite element tearing and interconnecting) algorithms with only Lagrange multipliers as unknowns, which has been developed by the engineering community, does not lead to a scalable algorithm with respect to the number of subdomains in each body. Instead, we consider a new method based on the saddle point formulation of the FETI methods with both displacement vectors and Lagrange multipliers as unknowns. The resulting system is solved with a block-diagonal preconditioner which combines the one-level FETI and the BDDC (balancing domain decomposition by constraints) methods. We show that this new method is scalable with respect to the number of subdomains. A model contact problem is solved by a nonlinear algorithm which combines the new method and a primal-dual active set method.

**Keywords** domain decomposition, scalable algorithms, FETI, BDDC, contact problems, primal-dual active set method

**AMS Subject Classification** 65F10, 65K10, 65N55

## 1 Introduction

We consider contact problems without friction. In [18, 20], we considered the FETI-FETI method, which is a domain decomposition method for a linearized contact problem. We showed that the FETI-FETI method, which has been used in the engineering community [2, 1], has a condition number estimate which grows linearly with the number of subdomains, or processors.

This paper<sup>1</sup> is a sequel to [18]; we introduce a scalable alternative to the FETI-FETI method, which we call a hybrid method. This method combines the one-level FETI and the BDDC methods.

In this paper, we assume the use of an active set method to solve our contact problem. In each step of an active set method, the active set is updated, and a minimization problem on the current active set is approximately solved, until a desired accuracy is achieved. Thus, an active set method requires (nonlinear) outer iterations in which the active set is updated and (linear) inner iterations in which a minimization problem is solved. We use the primal-dual active set strategy viewed as a semismooth Newton method, proposed and analyzed in [3, 4, 10, 9], to determine the outer iterations.

This paper is organized as follows. In Section 2, we review the one-level FETI, FETI-DP (dual-primal FETI), and BDDC methods. We introduce a nonlinear model problem and briefly describe the

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<sup>1</sup>An earlier version of the material contained in this paper already appeared in the Ph.D. thesis of the author, see [20].

FETI-FETI method in Section 3. We introduce the hybrid method and provide an eigenvalue analysis of its preconditioned operator in Sections 4 and 5. In Section 6 we present numerical results which confirm the scalability of the hybrid method. In Section 7, we solve a model problem using a combination of a primal-dual active set method and our hybrid method.

## 2 Building blocks of the Hybrid method

In this section, we briefly review some popular domain decomposition methods with nonoverlapping subdomains.

### 2.1 A model problem and notation

We consider a second-order scalar elliptic problem on a bounded domain  $\Omega \subset \mathbb{R}^n, n = 2, 3$ . We denote the boundary of  $\Omega$  by  $\partial\Omega$ , and assume that homogeneous Dirichlet boundary conditions are imposed on  $\partial\Omega_D \subset \partial\Omega$ , which is a subset of  $\partial\Omega$  with a positive measure. Let  $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_D$  be its complement. The corresponding Sobolev space in which the solution will be found is  $H_0^1(\Omega, \partial\Omega_D) := \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}$ . We find  $u \in H_0^1(\Omega, \partial\Omega_D)$  such that

$$a(u, v) = f(v), \quad \forall v \in H_0^1(\Omega, \partial\Omega_D), \quad (2.1)$$

where

$$a(u, v) := \int_{\Omega} \rho(x) \nabla u \cdot \nabla v, \quad f(v) = \int_{\Omega} f v. \quad (2.2)$$

Note that (2.1) is equivalent to the following minimization problem:

$$\min_{u \in H_0^1(\Omega, \partial\Omega_D)} \frac{1}{2} a(u, u) - f(u). \quad (2.3)$$

We decompose  $\Omega$  into  $N$  nonoverlapping subdomains  $\Omega_i, i = 1, \dots, N$ , each of which is the union of shape-regular elements with the finite element nodes on the boundaries of neighboring subdomains matching across the interface  $\Gamma := (\cup_{i=1}^N \partial\Omega_i) \setminus \partial\Omega_D$ .  $\Gamma$  is the union of

- faces, edges and vertices in three dimensions: faces, regarded as open subsets of  $\Gamma$ , are shared by two subdomains. Edges, regarded as open subsets of the boundaries of the faces, are shared by more than two subdomains. Vertices are endpoints of edges.
- edges and vertices in two dimensions: edges, regarded as open subsets of  $\Gamma$ , are shared by two subdomains. Vertices, as in three dimensions, are endpoints of edges.

We assume that  $\rho(x) = \rho_i \geq \rho_{min} > 0, \forall x \in \Omega_i, i = 1, \dots, N$ . We also introduce the corresponding set of interface nodes  $\Gamma_h := (\cup_{i=1}^N \partial\Omega_{i,h}) \setminus \partial\Omega_h$ , where  $\partial\Omega_h$  and  $\partial\Omega_{i,h}$  are the sets of finite element nodes on  $\partial\Omega$  and  $\partial\Omega_i$ , respectively. We also define local bilinear forms and linear functionals,

$$a^{(i)}(u, v) := \int_{\Omega_i} \rho(x) \nabla u \cdot \nabla v, \quad f^{(i)}(v) = \int_{\Omega_i} f v. \quad (2.4)$$

In the rest of this section, we discuss the choice of the space of finite element functions in one-level FETI, FETI-DP, and BDDC methods. We denote a standard finite element space of continuous, piecewise linear functions on  $\Omega_i$  by  $W^{(i)}$ . We will always assume that these functions vanish on  $\partial\Omega_D$ . Each  $W^{(i)}$  is decomposed into a subdomain interior part  $W_I^{(i)}$  and a subdomain interface part  $W_{\Gamma}^{(i)}$ :

$$W^{(i)} = W_I^{(i)} \oplus W_{\Gamma}^{(i)}.$$

We denote the associated product spaces by  $W := \prod_{i=1}^N W^{(i)}, W_I := \prod_{i=1}^N W_I^{(i)}$ , and  $W_{\Gamma} := \prod_{i=1}^N W_{\Gamma}^{(i)}$ .

The functions in  $W$  and  $W_{\Gamma}$  are in general discontinuous across the interface, whereas the finite element solutions are continuous across the interface  $\Gamma$ . Therefore we introduce continuous subspaces

of  $W$  and  $W_\Gamma$  by  $\widehat{W}$  and  $\widehat{W}_\Gamma$ , respectively.

For the FETI-DP and BDDC methods, we will also need a subspace  $\widetilde{W} \subset W$ , intermediate between  $W$  and  $\widehat{W}$ , which consists of finite element functions which satisfy certain continuity constraints. The corresponding interface space is denoted by  $\widetilde{W}_\Gamma$ .

We introduce the following decomposition of  $\widetilde{W}_\Gamma$ :

$$\widetilde{W}_\Gamma = W_\Delta \oplus \widehat{W}_\Pi = \left( \prod_{i=1}^N W_\Delta^{(i)} \right) \oplus \widehat{W}_\Pi,$$

where  $\widehat{W}_\Pi$  is a subspace of continuous functions and  $W_\Delta^{(i)}$  is a subspace of functions which are allowed to be discontinuous across the interface. More precisely,  $\widehat{W}_\Pi$  is spanned by subdomain vertex nodal basis functions, i.e., consists of functions which are nonzero only at subdomain vertices, in the two-dimensional case. Accordingly,  $W_\Delta^{(i)} \in W_\Gamma^{(i)}$  consist of functions which are zero at the vertices of the subdomain  $\Omega_i$ . In other words,  $\widetilde{W}_\Gamma$  consists of functions that are continuous at subdomain vertices. In the three-dimensional case, such vertex constraints are not enough to ensure scalability and we need to enforce the edge averages to be continuous; for instance, see [16]. See Figure 1 for a depiction of  $W$ ,  $\widetilde{W}$ , and  $\widehat{W}$  in the two-dimensional case.

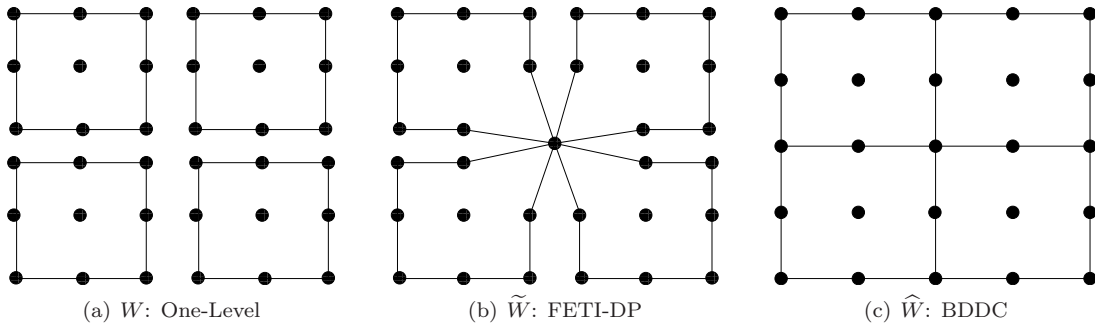
For each subdomain  $\Omega_i, i = 1, \dots, N$ , we assemble local stiffness matrices  $A^{(i)}$  and local load vectors  $f^{(i)}$  obtained by integrating appropriate expressions over individual subdomains.

We also introduce scaling factors  $\delta_i^\dagger(x)$  for each node  $x \in \Gamma_h \cap \partial\Omega_{i,h}, i = 1, \dots, N$ : for  $\gamma \in [1/2, \infty)$ ,

$$\delta_i^\dagger(x) = \frac{\rho_i^\gamma}{\sum_{j \in \mathcal{N}_x} \rho_j^\gamma}, \quad x \in \partial\Omega_{i,h} \cap \Gamma_h.$$

Here,  $\mathcal{N}_x$  is the set of indices  $j$  of the subdomains such that  $x \in \partial\Omega_{j,h}$ .

Figure 1:  $W$ ,  $\widetilde{W}$  and  $\widehat{W}$



## 2.2 The One-Level FETI method

In this subsection, we review the one-level FETI method, following [26, Section 6.3]. We use the finite element functions in the space  $W$  to discretize the minimization problem (2.3). Since the functions in  $W$  are in general discontinuous across the interface  $\Gamma$ , we need to enforce the continuity condition explicitly:

$$\min_{u \in W} \frac{1}{2} u^T A u - f^T u, \quad \text{subject to } B u = 0, \quad (2.5)$$

where

$$A = \begin{bmatrix} A^{(1)} & & \\ & \ddots & \\ & & A^{(N)} \end{bmatrix}, \quad f = \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(N)} \end{bmatrix}.$$

$Bu = 0$  represents continuity constraints across the interface  $\Gamma$ , where

$$B = [B^{(1)}, B^{(2)}, \dots, B^{(N)}]$$

is a matrix consisting of elements  $0, 1, -1$  such that  $Bu = 0$  if and only if all the values of  $u$  associated with more than one subdomain boundary coincide. The columns of  $B^{(i)}$  which correspond to the interior nodes of  $\Omega_i$  are zero. Thus,  $B^{(i)} = [0 \quad B_\Gamma^{(i)}]$  when the interior degrees of freedom are ordered first. We call  $B$  a jump operator. Introducing a vector of Lagrange multipliers  $\lambda$  to enforce the continuity constraint  $Bu = 0$ , we obtain the following Karush-Kuhn-Tucker (KKT) system:

Find  $(u, \lambda) \in W \times \text{range}(B)$ , such that

$$\begin{aligned} Au + B^T \lambda &= f \\ Bu &= 0 \end{aligned} \quad (2.6)$$

$\lambda$  is unique only up to an additive element of  $\ker(B^T)$ . The space of Lagrange multipliers,  $U$ , is therefore chosen as  $\text{range}(B)$ .

Eliminating the interior unknowns in each subdomain, we obtain the following:

Find  $(u_\Gamma, \lambda) \in W_\Gamma \times \text{range}(B_\Gamma)$ , such that

$$\begin{aligned} Su_\Gamma + B_\Gamma^T \lambda &= g \\ B_\Gamma u_\Gamma &= 0 \end{aligned} \quad (2.7)$$

where

$$S = \begin{bmatrix} S^{(1)} & & \\ & \ddots & \\ & & S^{(N)} \end{bmatrix}, \quad S^{(i)} = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)-1} A_{I\Gamma}^{(i)T}, \quad i = 1, \dots, N,$$

$$g = \begin{bmatrix} g^{(1)} \\ \vdots \\ g^{(N)} \end{bmatrix}, \quad g^{(i)} = f_\Gamma^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)-1} f_I^{(i)}, \quad i = 1, \dots, N$$

and  $B_\Gamma = [B_\Gamma^{(1)}, B_\Gamma^{(2)}, \dots, B_\Gamma^{(N)}]$  is obtained by removing the zero columns of  $B$  that correspond to the interior nodes of individual subdomains, resulting in  $Bu = B_\Gamma u_\Gamma$  where  $B = [0 \quad B_\Gamma]$  and  $u^T = [u_I^T \quad u_\Gamma^T]$ .

In all FETI methods, we reduce the KKT system (2.7) to an equation of  $\lambda$  alone, by solving the first equation of (2.7) for  $u_\Gamma$ . The matrices  $A$  in (2.6) and  $S$  in (2.7), however, are generally singular, when there are subdomains with boundaries which do not intersect the Dirichlet boundary  $\partial\Omega_D$ . We call such subdomains floating. In such a case the solution of the first equation of (2.7) exists if and only if  $g - B_\Gamma^T \lambda \in \text{range}(S)$ ; this requirement leads to the introduction of a projection  $P$ , which will be introduced shortly. First, we introduce a matrix  $R$  such that  $\text{range}(R) = \ker(S)$ :

$$R = \begin{bmatrix} R^{(1)} & & \\ & \ddots & \\ & & R^{(N)} \end{bmatrix},$$

where  $R^{(i)}$  consists of the null vectors of  $S^{(i)}$ ,  $i = 1, \dots, N$ . Subdomains with nonsingular stiffness matrices do not contribute to the matrix  $R$ , i.e.,  $R^{(i)}$  is an empty matrix if the subdomain  $\Omega_i$  intersects the Dirichlet boundary  $\partial\Omega_D$ . We now solve the first equation of (2.7) for  $u_\Gamma$ :

$$u_\Gamma = S^\dagger (g - B_\Gamma^T \lambda) + R\alpha \quad \text{if} \quad g - B_\Gamma^T \lambda \in \text{range}(S) = \ker(S)^\perp = \text{range}(R)^\perp, \quad (2.8)$$

where  $S^\dagger$  is a pseudoinverse of  $S$  and  $\alpha$  has to be determined. Substituting (2.8) into the second equation of (2.7), we obtain

$$B_\Gamma S^\dagger B_\Gamma^T \lambda = B_\Gamma S^\dagger g + B_\Gamma R \alpha. \quad (2.9)$$

We introduce the notation  $F := B_\Gamma S^\dagger B_\Gamma^T$ ,  $d := B_\Gamma S^\dagger g$ ,  $G := B_\Gamma R$ ,  $e := R^T g$  and  $P := I - G(G^T G)^{-1} G^T$ . Note that  $P$  is a projection operator with its range orthogonal to  $G$ . We apply this  $P$  to (2.9) to eliminate the term with  $\alpha$  and rewrite the orthogonality condition in (2.8) to obtain the following:

$$\begin{cases} PF\lambda &= Pd \\ G^T \lambda &= e. \end{cases} \quad (2.10)$$

We define the space

$$V := \{\mu \in U : B_\Gamma^T \mu \in \text{range}(S)\} = \ker(G^T),$$

which we call the space of admissible increments, following Chen and Mandel [7]. The one-level FETI method is a preconditioned conjugate gradient method applied to

$$PF\lambda = Pd, \quad \lambda \in \lambda_0 + V \quad (2.11)$$

where  $\lambda_0$  is chosen such that  $G^T \lambda_0 = e$ . Here, we only consider the *Dirichlet preconditioner*  $M_D^{-1} := B_{D,\Gamma} S B_{D,\Gamma}^T$ , where  $B_{D,\Gamma} = [B_{D,\Gamma}^{(1)} \cdots B_{D,\Gamma}^{(N)}]$  is a scaled jump operator.  $B_{D,\Gamma}^{(i)}$  is obtained as follows: each nonzero entry of  $B_\Gamma^{(i)}$  is related to the Lagrange multiplier enforcing the continuity at a node  $x \in \partial\Omega_i \cap \partial\Omega_j$  and is multiplied by  $\delta_j^\dagger(x)$  to produce the corresponding entry of  $B_{D,\Gamma}^{(i)}$ .

With this choice of preconditioner, the preconditioned operator of the one-level FETI method has the following condition number bound:

$$\mathcal{K} \leq C(1 + \log(H/h))^2, \quad (2.12)$$

where  $\mathcal{K}$  denotes the condition number of the preconditioned operator in the appropriate subspace. For a proof of (2.12), see [24] or [26, Section 6.3]. Thus the convergence rate of the one-level FETI method depends only polylogarithmically on the number of degrees of freedom of a subdomain.

### 2.3 The FETI-DP method

In this subsection, we closely follow the notation of [21]. For more details on various FETI-DP methods, see, e.g., [21, 16, 11, 12, 26] and the references therein. In the FETI-DP method, we use finite element functions in  $\widetilde{W}$  to discretize (2.3).

We first note that the local stiffness matrices  $A^{(i)}$  and the local load vectors  $f^{(i)}$  can be written as follows:

$$A^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)T} & A_{\Pi I}^{(i)T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Pi I}^{(i)T} \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{bmatrix}, \quad f^{(i)} = \begin{bmatrix} f_I^{(i)} \\ f_\Delta^{(i)} \\ f_\Pi^{(i)} \end{bmatrix}, \quad (2.13)$$

where  $I, \Delta$ , and  $\Pi$  indicate the index sets corresponding to the interior nodes, dual nodes, i.e., those of  $W_\Delta^{(i)}$ , and primal nodes, i.e., those of  $W_\Pi^{(i)}$ , respectively. We introduce the matrix  $\widetilde{A}$ , which can be thought of as the restriction of  $A$ , defined for the functions in  $W$ , to the subspace  $\widetilde{W}$ :

$$\widetilde{A} = \begin{bmatrix} A_{II}^{(1)} & A_{\Delta I}^{(1)T} & & & & \widetilde{A}_{\Pi I}^{(1)T} \\ A_{\Delta I}^{(1)} & A_{\Delta\Delta}^{(1)} & & & & \widetilde{A}_{\Pi\Delta}^{(1)T} \\ & & \ddots & & & \vdots \\ & & & A_{II}^{(N)} & A_{\Delta I}^{(N)T} & \widetilde{A}_{\Pi I}^{(N)T} \\ & & & A_{\Delta I}^{(N)} & A_{\Delta\Delta}^{(N)} & \widetilde{A}_{\Pi\Delta}^{(N)T} \\ \widetilde{A}_{\Pi I}^{(1)} & \widetilde{A}_{\Pi\Delta}^{(1)} & \cdots & \widetilde{A}_{\Pi I}^{(N)} & \widetilde{A}_{\Pi\Delta}^{(N)} & \widetilde{A}_{\Pi\Pi} \end{bmatrix}. \quad (2.14)$$

Here,

$$\tilde{A}_{\Pi I}^{(i)} = R_{\Pi}^{(i)T} A_{\Pi I}^{(i)}, \quad \tilde{A}_{\Pi \Delta}^{(i)} = R_{\Pi}^{(i)T} A_{\Pi \Delta}^{(i)}, \quad i = 1, \dots, N,$$

and

$$\tilde{A}_{\Pi \Pi} = \sum_{i=1}^N R_{\Pi}^{(i)T} A_{\Pi \Pi}^{(i)} R_{\Pi}^{(i)},$$

where  $R_{\Pi}^{(i)} : \widehat{W}_{\Pi} \rightarrow W_{\Pi}^{(i)}$ ,  $i = 1, \dots, N$ , is a restriction operator which extracts the relevant subdomain component belonging to  $W_{\Pi}^{(i)}$  from a vector in  $\widehat{W}_{\Pi}$ . As in the one-level FETI method, we introduce a vector of Lagrange multipliers and obtain the following saddle point problem:

Find  $(u, \lambda) \in \widetilde{W} \times \text{range}(\tilde{B})$ , such that

$$\begin{aligned} \tilde{A}u + \tilde{B}^T \lambda &= f \\ \tilde{B}u &= 0 \end{aligned} \quad (2.15)$$

Again,  $\tilde{B}$  is a jump operator such that  $\tilde{B}u = 0$ ,  $u \in \widetilde{W}$  if and only if the values of  $u$  associated with more than one subdomain coincide. Eliminating the interior unknowns of each subdomain from the system (2.15), we obtain:

Find  $(u, \lambda) \in \widetilde{W}_{\Gamma} \times \text{range}(\tilde{B}_{\Gamma})$ , such that

$$\begin{aligned} \tilde{S}_{\Gamma} u_{\Gamma} + \tilde{B}_{\Gamma}^T \lambda &= g \\ \tilde{B}_{\Gamma} u &= 0 \end{aligned} \quad (2.16)$$

$\tilde{S}_{\Gamma}$  can also be regarded as the restriction of  $S$ , defined on  $W_{\Gamma}$ , to the subspace  $\widetilde{W}_{\Gamma}$ :

$$\tilde{S}_{\Gamma} = \tilde{R}_{\Gamma}^T S \tilde{R}_{\Gamma},$$

where  $\tilde{R}_{\Gamma} : \widetilde{W}_{\Gamma} \rightarrow W_{\Gamma}$  is a direct sum of restriction operators that extract the subdomain part belonging to  $W_{\Gamma}^{(i)}$  from a vector in  $\widetilde{W}_{\Gamma}$ .

The matrix  $\tilde{A}$ , and therefore also  $\tilde{S}_{\Gamma}$ , are nonsingular, so we can solve the first equation of (2.16) for  $u_{\Gamma}$  and substitute the resulting expression into the second equation of (2.16):

$$\tilde{B}_{\Gamma} \tilde{S}_{\Gamma}^{-1} \tilde{B}_{\Gamma}^T \lambda = -\tilde{B}_{\Gamma} \tilde{S}_{\Gamma}^{-1} g. \quad (2.17)$$

The Dirichlet preconditioner used in the FETI-DP algorithms to solve the equation (2.17) is  $\tilde{B}_{D,\Gamma} \tilde{S}_{\Gamma} \tilde{B}_{D,\Gamma}^T$ .  $\tilde{B}_{D,\Gamma} = [\tilde{B}_{D,\Gamma}^{(1)}, \dots, \tilde{B}_{D,\Gamma}^{(N)}]$  is obtained in exactly the same manner as  $B_{D,\Gamma}$  in Section 2.2.

With this choice of preconditioner, the preconditioned operator for the FETI-DP method also has the condition number bound (2.12). For a proof of this convergence bound for the two-dimensional case, see, e.g., [25]. For three-dimensional scalar elliptic problems and linear elasticity problems, see [17] and [16], respectively.

## 2.4 The BDDC method

In this subsection, we review the BDDC method, following [27].

The discretized problem on the entire domain  $\Omega$  is:

Find  $(u_I, u_{\Gamma}) \in (W_I, \widetilde{W}_{\Gamma})$ , such that

$$\begin{pmatrix} A_{II} & A_{\Gamma I}^T \\ A_{\Gamma I} & A_{\Gamma \Gamma} \end{pmatrix} \begin{pmatrix} u_I \\ u_{\Gamma} \end{pmatrix} = \begin{pmatrix} f_I \\ f_{\Gamma} \end{pmatrix}. \quad (2.18)$$

The equation (2.18) can be rewritten as

$$\begin{bmatrix} A_{II}^{(1)} & & & A_{\Gamma I}^{(1)T} \widehat{R}_{\Gamma}^{(1)} \\ & \ddots & & \vdots \\ & & A_{II}^{(N)} & A_{\Gamma I}^{(N)T} \widehat{R}_{\Gamma}^{(N)} \\ \widehat{R}_{\Gamma}^{(1)T} A_{\Gamma I}^{(1)} & \dots & \widehat{R}_{\Gamma}^{(N)T} A_{\Gamma I}^{(N)} & \sum_{i=1}^N \widehat{R}_{\Gamma}^{(i)T} A_{\Gamma \Gamma}^{(i)} \widehat{R}_{\Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} u_I^{(1)} \\ \vdots \\ u_I^{(N)} \\ u_{\Gamma} \end{bmatrix} = \begin{bmatrix} f_I^{(1)} \\ \vdots \\ f_I^{(N)} \\ \sum_{i=1}^N R_{\Gamma}^{(i)T} f_{\Gamma}^{(i)} \end{bmatrix}, \quad (2.19)$$

where  $\widehat{R}_\Gamma^{(i)} : \widehat{W}_\Gamma \rightarrow W_\Gamma^{(i)}$  are restriction operators which extract the subdomain parts. Eliminating the interior unknowns of each subdomain, i.e., eliminating the upper left block of (2.19), we obtain

$$\widehat{S}_\Gamma u_\Gamma = g_\Gamma, \quad (2.20)$$

where

$$\begin{aligned} \widehat{S}_\Gamma &= \sum_{i=1}^N \widehat{R}_\Gamma^{(i)T} (A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)-1} A_{\Gamma I}^{(i)T}) \widehat{R}_\Gamma^{(i)} \\ &= \sum_{i=1}^N \widehat{R}_\Gamma^{(i)T} S^{(i)} \widehat{R}_\Gamma^{(i)} \\ &= \widehat{R}_\Gamma^T S \widehat{R}_\Gamma, \end{aligned} \quad (2.21)$$

and

$$g_\Gamma = \sum_{i=1}^N \widehat{R}_\Gamma^{(i)T} (f_\Gamma^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)-1} f_I^{(i)}).$$

$\widehat{R}_\Gamma : \widehat{W}_\Gamma \rightarrow W_\Gamma$  and  $S$  are direct sums of  $\widehat{R}_\Gamma^{(i)}$  and  $S^{(i)}$ , respectively. From (2.21), we can see that  $\widehat{S}_\Gamma$  can be regarded as the restriction of  $S$ , defined on  $W_\Gamma = \prod_{i=1}^N W_\Gamma^{(i)}$ , to the continuous subspace  $\widehat{W}_\Gamma$ . We can also view  $\widehat{S}_\Gamma$  as the restriction of  $\widetilde{S}_\Gamma$  to  $\widehat{W}_\Gamma$ :

$$\widehat{S}_\Gamma = \bar{R}_\Gamma^T \widetilde{S}_\Gamma \bar{R}_\Gamma,$$

where  $\bar{R}_\Gamma : \widehat{W}_\Gamma \rightarrow \widetilde{W}_\Gamma$ . We introduce a few more restriction operators that constitute  $\bar{R}_\Gamma$ .  $R_{\Gamma\Delta}^{(i)} : \widehat{W}_\Gamma \rightarrow W_\Delta^{(i)}$  extracts the part that belongs to  $W_\Delta^{(i)}$  from a vector in  $\widehat{W}_\Gamma$ .  $R_{\Gamma\Pi} : \widehat{W}_\Gamma \rightarrow \widehat{W}_\Pi$  is defined similarly. Thus  $\bar{R}_\Gamma = [R_{\Gamma\Delta}^{(1)T} \cdots R_{\Gamma\Delta}^{(N)T} R_{\Gamma\Pi}^T]$ . We also define  $R_{D,\Gamma\Delta}^{(i)}$ , which are scaled versions of  $R_{\Gamma\Delta}^{(i)}$ ; each row of  $R_{\Gamma\Delta}^{(i)}$  has exactly one nonzero entry corresponding to a node  $x$  on the subdomain interface. Multiplying each such entry with  $\delta_i^\dagger(x)$  results in the scaled version  $R_{D,\Gamma\Delta}^{(i)}$ . In the BDDC method, we use  $M^{-1} = \bar{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \bar{R}_{D,\Gamma}$  as the preconditioner.

With this choice of preconditioner, the BDDC and the FETI-DP algorithms have the same set of eigenvalues; see [22, 23, 21].

## 3 A Model Problem

### 3.1 Motivation and Notations

Our ultimate goal is to solve multi-body contact problems without friction. Contact problems are characterized by an active area of contact, which is unknown a priori, and inequality constraints such as non-penetration conditions; e.g., see [2, 1].

We consider two slightly different approaches to derive the same saddle point formulation we are going to solve. In the first approach the method of attack is an active set method from the outset, in which we solve a sequence of equality constrained minimization problems. We choose to solve a saddle point formulation of each such equality constrained problem for the reason we will explain below.

In the second approach, described e.g. in [9], a complementarity problem of the original minimization problem is first considered. This problem can be expressed as a single nonlinear equation, and we can use a semismooth Newton method to solve this nonlinear problem. It turns out that the linear problems that are solved in this approach are identical to the saddle point problems of the first approach, and the two approaches differ only in the solution update process.

In this paper, we develop the theory only for scalar elliptic problems with inequality constraints and present the following model problem, which is taken from [5, 6], as a motivation:

$$\min \sum_{i=1}^2 \left( \frac{1}{2} \int_{\Omega^i} |\nabla u^i|^2 dx - \int_{\Omega^i} f u^i dx \right)$$

$$\begin{aligned}
\text{where } \quad & u^i \in H^1(\Omega^i), i = 1, 2, \quad \Omega^1 = (0, 1) \times (0, 1), \Omega^2 = (1, 2) \times (0, 1) \\
& u^1 = 0 \quad \text{on} \quad \Gamma_u^1 = \{0\} \times (0, 1) \\
& u^2 - u^1 \geq 0 \quad \text{on} \quad \Gamma_c = \{1\} \times (0, 1)
\end{aligned} \tag{3.1}$$

The reason we consider only scalar elliptic problems is that the inequality constraints in scalar elliptic problems are much simpler than those in linear elasticity problems and this simplicity allows us to focus on the analysis of the preconditioned operator.

We consider a generalization of (3.1), a minimization problems with multiple bodies  $\Omega_i, i = 1, \dots, N$ , each of which has many degrees of freedom and is decomposed into subdomains  $\Omega_{i,j}, i = 1, \dots, N, j = 1, \dots, N_i$ , constrained by an inequality condition:

$$\begin{aligned}
\min \quad & \sum_{i=1}^N \left( \frac{1}{2} \int_{\Omega^i} \rho |\nabla u^i|^2 dx - \int_{\Omega^i} f u^i dx \right) \\
\text{where} \quad & u^i \in H^1(\Omega^i), \quad i = 1, \dots, N, \\
& u^i = 0 \quad \text{on} \quad \Gamma_u^i, \\
& \sum_{i=1}^N B^{(i)} u^i \leq 0
\end{aligned} \tag{3.2}$$

We assume that the boundary of at least one body is clamped, i.e.,  $\partial\Omega_D := \cup_{i=1}^N \Gamma_u^i \neq \emptyset$ . We assume that there are no traction forces. We assume that  $\rho(x) = \rho_{i,j} \geq \rho_{min} > 0, \forall x \in \Omega_{i,j}, \forall i, j$ . We also assume the existence of a coefficient  $C \geq 1$ , independent of  $i$ , such that

$$\rho_i := \max_j \rho_{i,j} \leq C \rho_{i,j}, \quad \forall i, j. \tag{3.3}$$

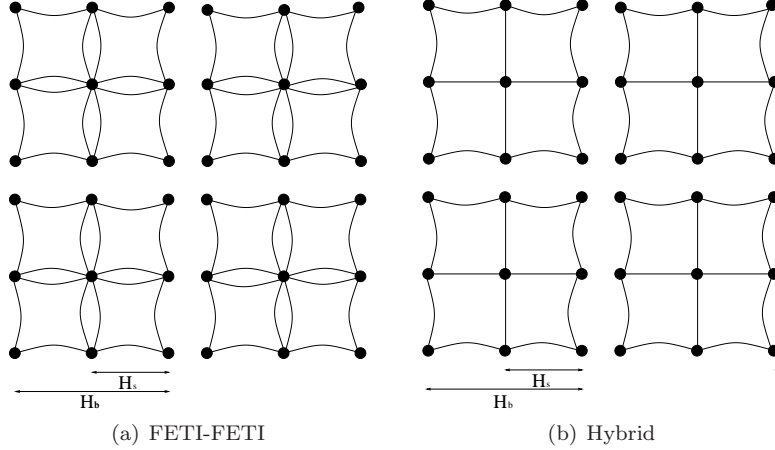
We introduce two types of global interfaces: the first one is  $\overline{\Gamma}_{gl} := \cup_{i \neq j} \partial\Omega_i \cap \partial\Omega_j$ , and can be viewed as the potential contact area between the bodies: in the model problem, this is  $\Gamma_c$ . The second one, the *current* contact area, is denoted by  $\Gamma_{gl}^k$ , where  $\Gamma_{gl}^k \subset \overline{\Gamma}_{gl}$ ; the superscript  $k$  concerns the outer iteration of the active set method and reminds us that the current active set/area changes. In each outer iteration of the active set method some of the inequality constraints are adopted as the corresponding equality constraints and the rest are ignored, and  $\Gamma_{gl,h}^k$ , the discrete version of  $\Gamma_{gl}^k$ , can be viewed as the collection of the nodes at which equality constraints are being imposed. We also introduce the local interfaces  $\Gamma_{loc}^{(i)} := \cup_{j \neq k} (\partial\Omega_{i,j} \cap \partial\Omega_{i,k}), i = 1, \dots, N$ . We denote the union of the free boundaries by  $\partial\Omega_F := (\cup_i \partial\Omega_i) \setminus (\partial\Omega_D \cup \overline{\Gamma}_{gl})$ .

We denote the standard finite element space of continuous, piecewise linear functions on  $\Omega_{i,j}$  by  $W^{(i,j)}$ . Each  $W^{(i,j)}$  is decomposed into a subdomain interior part  $W_I^{(i,j)}$  and a subdomain interface part  $W_\Gamma^{(i,j)}$  for functions on  $\partial\Omega_{i,j} \cap \overline{\Gamma}_{gl}$ . We also recall that all functions vanish on the Dirichlet boundary  $\partial\Omega_D$ . We define associated product spaces,  $W_I^{(i)} := \prod_{j=1}^{N_i} W_I^{(i,j)}$  and  $W_\Gamma^{(i)} := \prod_{j=1}^{N_i} W_\Gamma^{(i,j)}$ . We introduce spaces analogous to  $\widetilde{W}_\Gamma$  and  $\widehat{W}_\Gamma$  of Section 2. Functions in  $W_\Gamma^{(i)}$  are in general discontinuous across the local interface  $\Gamma_{loc}^{(i)}$ , and we define  $\widehat{W}_\Gamma^{(i)}$  as the continuous subspace of  $W_\Gamma^{(i)}$ .  $\widetilde{W}_\Gamma^{(i)} := W_\Delta^{(i)} \oplus \widehat{W}_\Pi^{(i)}$  is an intermediate space between  $\widehat{W}_\Gamma^{(i)}$  and  $W_\Gamma^{(i)}$ . In the two-dimensional case, which is the focus of our analysis in this paper,  $\widetilde{W}_\Gamma^{(i)}$  consists of functions that are continuous at subdomain vertices. Also, we let  $\widetilde{W}^{(i)} := W_I^{(i)} \oplus \widetilde{W}_\Gamma^{(i)}$ .

We introduce the matrices  $A^{(i)}$ , which are the direct sums of the stiffness matrices  $A^{(i,j)}, j = 1, \dots, N_i$  for the individual subdomains:

$$A^{(i)} = \begin{bmatrix} A^{(i,1)} & & \\ & \ddots & \\ & & A^{(i,N_i)} \end{bmatrix}, i = 1, \dots, N. \tag{3.4}$$

Figure 2:  $\widetilde{W}_c$  for FETI-FETI,  $\widehat{W}_c$  for Hybrid



We also introduce the Schur complements

$$S^{(i)} = \begin{bmatrix} S^{(i,1)} & & \\ & \ddots & \\ & & S^{(i,N_i)} \end{bmatrix}, \quad S^{(i,j)} = A_{\Gamma\Gamma}^{(i,j)} - A_{\Gamma I}^{(i,j)T} A_{II}^{(i,j)-1} A_{\Gamma I}^{(i,j)T}, j = 1, \dots, N_i.$$

The finite element formulation of the problem (3.1) in the space  $\widetilde{W}_c := \prod_{i=1}^N \widetilde{W}^{(i)}$  and  $\widetilde{W}_{\Gamma,c} := \prod_{i=1}^N \widetilde{W}_{\Gamma}^{(i)}$  has been considered in the FETI-FETI method [20, 18]. In the hybrid method, we formulate the problem in the space  $\widehat{W}_{\Gamma,c} := \prod_{i=1}^N \widehat{W}_{\Gamma}^{(i)}$ . We will briefly describe the FETI-FETI method here. The problem (3.1) can be expressed as

$$\min_{u \in \widetilde{W}_c} \frac{1}{2} u^T \widetilde{A}_c u - \widetilde{f}_c^T u, \quad \text{with } \widetilde{B}_c u \leq 0. \quad (3.5)$$

Recalling that we are using an active set method to deal with the inequality conditions, we formulate the minimization problem on the current active set:

$$\min_{u \in \widetilde{W}_c} \frac{1}{2} u^T \widetilde{A}_c u - \widetilde{f}_c^T u, \quad \text{with } Z^k \widetilde{B}_c u = 0, \quad (3.6)$$

where

$$\widetilde{A}_c = \begin{bmatrix} \widetilde{A}^{(1)} & & \\ & \ddots & \\ & & \widetilde{A}^{(N)} \end{bmatrix}, u = \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(N)} \end{bmatrix}, \widetilde{f}_c = \begin{bmatrix} \widetilde{f}^{(1)} \\ \vdots \\ \widetilde{f}^{(N)} \end{bmatrix}, u^{(i)} \in \widetilde{W}^{(i)}, i = 1, \dots, N,$$

and

$$\widetilde{B}_c = \begin{bmatrix} B_{loc} \\ B_{gl} \end{bmatrix} = \begin{bmatrix} B_{loc}^{(1)} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & B_{loc}^{(N)} \\ B_{gl}^{(1)} & \cdots & B_{gl}^{(N)} \end{bmatrix}, \quad B_{loc}^{(i)} = \begin{bmatrix} B_{loc}^{(i,1)} & \cdots & B_{loc}^{(i,N_i)} \end{bmatrix}, i = 1, \dots, N,$$

$$Z^k = \begin{bmatrix} I & 0 \\ 0 & Z_{gl}^k \end{bmatrix}.$$

$Z^k \tilde{B}_c u = 0$  in (3.6) indicates the continuity constraint across the local subdomain interface  $\Gamma_{loc}^{(i)}$ ,  $i = 1, \dots, N$ , as well as the continuity constraint across the global area of contact  $\Gamma_{gl}^k$ .  $Z_{gl}^k$  is a square matrix obtained by replacing some of the diagonal entries of the identity matrix with zeros; only the entries corresponding to the nodes at which an equality is being imposed are retained. We use the superscript  $k$  as an indication that  $Z_{gl}^k$  and  $Z^k$  change in each iteration of the active set method. We have  $B_{loc}^{(i)} u^{(i)} = 0$ ,  $u \in \tilde{W}^{(i)}$ , exactly when the values associated with more than one subdomain on the body  $\Omega_i$  coincide. Note that  $B_{loc}^{(i)}$  has nonzero columns only for the components of  $W_{\Delta}^{(i)}$ . We also introduce a scaled jump operator,  $\tilde{B}_{D,c}$ :

$$\tilde{B}_{D,c} = \begin{bmatrix} B_{loc,D} \\ B_{gl,D} \end{bmatrix} = \begin{bmatrix} B_{loc,D}^{(1)} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & B_{loc,D}^{(N)} \\ B_{gl,D}^{(1)} & \cdots & B_{gl,D}^{(N)} \end{bmatrix}$$

and

$$B_{loc,D}^{(i)} = \begin{bmatrix} B_{loc,D}^{(i,1)} & \cdots & B_{loc,D}^{(i,N_i)} \end{bmatrix}, i = 1, \dots, N.$$

$B_{loc,D}^{(i)}$  and  $B_{gl,D}^{(i)}$  are obtained in the same manner as the  $B_{D,\Gamma}$  of the one-level FETI method (see section 2.2); the nonzero entry of  $B_{loc}^{(i,j)}$  associated with the Lagrange multipliers for the continuity at the node  $x \in \partial\Omega_{i,j} \cap \Omega_{i,k}$ , multiplied by  $\delta_{i,k}^\dagger(x) = \rho_{i,k}^\gamma(x) / \sum_{s \in \mathcal{N}_{x,loc}^{(i)}} \rho_{i,s}^\gamma(x)$ , where  $\mathcal{N}_{x,loc}^{(i)}$  is the set of indices of the subdomains of  $\Omega_i$  with  $x$  on their boundary, is the corresponding entry of  $B_{loc,D}^{(i)}$ . Similarly, the nonzero entry of  $B_{gl}^{(i)}$  associated with the Lagrange multiplier for the continuity at the node  $x \in \partial\Omega_i \cap \partial\Omega_j$ , multiplied by  $\delta_j^\dagger(x) = \sum_{s \in \mathcal{N}_{x,loc}^{(j)}} \rho_{j,s}^\gamma(x) / \sum_{k \in \mathcal{N}_{x,gl}, t \in \mathcal{N}_{x,loc}^{(k)}} \rho_{k,t}^\gamma(x)$ , where  $\mathcal{N}_{x,gl}$  is the set of indices of the subdomains of any body which share the node  $x$  on their boundary, is the corresponding entry of  $B_{gl,D}^{(i)}$ .

Eliminating the interior unknowns in all subdomains of each body, we obtain the following reduced minimization problem,

$$\min_{u_\Gamma \in \tilde{W}_{\Gamma,c}} \frac{1}{2} u_\Gamma^T \tilde{S}_c u_\Gamma - \tilde{g}_c^T u_\Gamma, \quad \text{with} \quad Z^k \tilde{B}_{\Gamma,c} u_\Gamma = 0, \quad (3.7)$$

where

$$\tilde{S}_c = \begin{bmatrix} \tilde{S}_\Gamma^{(1)} & & \\ & \ddots & \\ & & \tilde{S}_\Gamma^{(N)} \end{bmatrix}, u_\Gamma = \begin{bmatrix} u_\Gamma^{(1)} \\ \vdots \\ u_\Gamma^{(N)} \end{bmatrix}, \quad u_\Gamma^{(i)} \in \tilde{W}_\Gamma^{(i)}, i = 1, \dots, N.$$

This minimization problem has the following KKT system:

$$\begin{bmatrix} \tilde{S}_c & (Z^k \tilde{B}_{\Gamma,c})^T \\ Z^k \tilde{B}_{\Gamma,c} & 0 \end{bmatrix} \begin{bmatrix} u_\Gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix}. \quad (3.8)$$

It is natural to reduce this system to an equation for  $\lambda$  as in the one-level FETI method and solve it with the PCG method in a proper subspace, using the following preconditioner:

$$M_D^{-1} := Z^k \tilde{B}_{D,\Gamma_c} \tilde{S}_c \tilde{B}_{D,\Gamma_c}^T Z^k.$$

The resulting method, i.e., the FETI-FETI method, turns out not to be scalable with respect to the number of subdomains; see [18, 20]. We now present a scalable alternative, which we name a hybrid method.

## 4 A Hybrid Method

The hybrid method introduced in this section is a scalable alternative to the FETI-FETI method. We use finite element functions in the space  $\widehat{W}_{\Gamma,c} := \prod_{i=1}^N \widehat{W}_{\Gamma}^{(i)}$  to discretize the contact problem.  $W_{\Gamma_{OL}}^{(i)}$  denotes a finite element space on  $\partial\Omega_i \cap \overline{\Gamma_{gl}}$ . Here, *OL* stands for one-level; this is because we will use one-level FETI type preconditioners, which is obtained by regarding each body as a single subdomain without further decomposition.

We introduce the Schur complement  $\widehat{S}_{\Gamma}^{(i)}$  on  $\widehat{W}_{\Gamma}^{(i)}$ , which can be obtained by restricting  $S^{(i)}$  to  $\widehat{W}_{\Gamma}^{(i)}$ :

$$\widehat{S}_{\Gamma}^{(i)} = \widehat{R}_{\Gamma}^{(i)T} S^{(i)} \widehat{R}_{\Gamma}^{(i)}, i = 1, \dots, N,$$

where  $\widehat{R}_{\Gamma}^{(i)} : \widehat{W}_{\Gamma}^{(i)} \rightarrow W_{\Gamma}^{(i)}$ . We also introduce restriction operators  $\bar{R}_{\Gamma}^{(i)} : \widehat{W}_{\Gamma}^{(i)} \rightarrow \widetilde{W}_{\Gamma}^{(i)}$ ,

$$\bar{R}_{\Gamma}^{(i)} = \begin{bmatrix} R_{\Gamma\Delta}^{(i,1)} \\ \vdots \\ R_{\Gamma\Delta}^{(i,N_i)} \\ R_{\Gamma\Pi}^{(i)} \end{bmatrix},$$

where  $R_{\Gamma\Delta}^{(i,j)}$  extracts from a vector in  $\widehat{W}_{\Gamma}^{(i)}$  the part that belongs to  $W_{\Delta}^{(i,j)}$  and  $R_{\Gamma\Pi}^{(i)} : \widehat{W}_{\Gamma}^{(i)} \rightarrow \widehat{W}_{\Pi}^{(i)}$  is defined similarly. We also define the scaled versions  $\bar{R}_{D,\Gamma}^{(i)}$ :

$$\bar{R}_{D,\Gamma}^{(i)} = \begin{bmatrix} R_{D,\Gamma\Delta}^{(i,1)} \\ \vdots \\ R_{D,\Gamma\Delta}^{(i,N_i)} \\ R_{\Gamma\Pi}^{(i)} \end{bmatrix}.$$

Here,  $R_{D,\Gamma\Delta}^{(i,j)}$  is obtained as follows: a nonzero entry of  $R_{\Gamma\Delta}^{(i,j)}$ , which corresponds to a node  $x \in \partial\Omega_{i,j,h} \setminus \partial\Omega_{i,h}$ , is multiplied by  $\delta_{i,j}^{\dagger}(x)$ , where

$$\delta_{i,j}^{\dagger}(x) := \frac{\rho_{i,j}^{\gamma}(x)}{\sum_{k \in \mathcal{N}_{x,loc}^{(i)}} \rho_{i,k}^{\gamma}(x)}.$$

The restriction of the minimization problem (3.7) in  $\widetilde{W}_{\Gamma,c}$  to the subspace  $\widehat{W}_{\Gamma,c}$  is as follows:

$$\min_{u_{\Gamma} \in \widehat{W}_{\Gamma,c}} \frac{1}{2} u_{\Gamma}^T \widehat{S}_c u_{\Gamma} - \widehat{g}_c^T u_{\Gamma}, \quad \text{with} \quad \widehat{Z}^k \widehat{B}_{\Gamma,c} u_{\Gamma} = 0, \quad (4.1)$$

where

$$\widehat{S}_c = \begin{bmatrix} \widehat{S}_{\Gamma}^{(1)} & & \\ & \ddots & \\ & & \widehat{S}_{\Gamma}^{(N)} \end{bmatrix},$$

and where  $\widehat{B}_{\Gamma,c}$  only retains rows and columns of  $\widetilde{B}_{\Gamma,c}$  which correspond to  $\widehat{W}_{\Gamma,c}$ , and  $\widehat{Z}^k$  is obtained by removing irrelevant columns and rows of  $Z^k$ .

In addition to  $\widehat{B}_{\Gamma,c}$ , we define another jump operator  $B_{\Gamma_{OL}}$  which acts on vectors of the space  $\prod_{i=1}^N W_{\Gamma_{OL}}^{(i)}$ . This operator is needed in the preconditioner for the hybrid method. Recall that  $\widehat{B}_{\Gamma,c}$  acts on vectors of the space  $\widehat{W}_{\Gamma,c}$  and only has rows corresponding to the Lagrange multipliers enforcing the continuity between different bodies, i.e., continuity across  $\overline{\Gamma_{gl}}$ . Thus  $B_{\Gamma_{OL}}$  and  $\widehat{B}_{\Gamma,c}$  differ only in that  $\widehat{B}_{\Gamma,c}$  has a number of zero columns which correspond to the nodes on  $\Gamma_{loc,h}$ . We note that  $B_{\Gamma_{OL}}$  can be regarded as the jump operator for the one-level FETI method resulting from viewing each body as

a subdomain. We can define the scaled jump operator  $B_{\Gamma_{OL},D}$  in the usual way. Introducing a vector of Lagrange multipliers  $\lambda$ , we arrive at the following saddle point formulation of (4.1):

$$\text{Find } (u_\Gamma, \lambda) \in \widehat{W}_{\Gamma,c} \times \text{range}(\widehat{B}_{\Gamma,c})$$

$$\begin{bmatrix} \widehat{S}_c & (\widehat{Z}^k \widehat{B}_{\Gamma,c})^T \\ \widehat{Z}^k \widehat{B}_{\Gamma,c} & 0 \end{bmatrix} \begin{bmatrix} u_\Gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} \widehat{g}_c \\ 0 \end{bmatrix}. \quad (4.2)$$

We can solve (4.2) by reducing the system to an equation of  $\lambda$  alone in a proper subspace, but that requires inverting  $\widehat{S}_c$ , which is expensive. Instead, we keep the saddle point problem (4.2) as is and solve it by a Krylov subspace method which can deal with indefinite systems, such as the preconditioned conjugate residual (PCR) method. Due to the singularity of the matrix  $\widehat{S}_c$ , the solution of the upper part of the system (4.2) exists if and only if  $\widehat{g}_c - (\widehat{Z}^k \widehat{B}_{\Gamma,c})^T \lambda \in \text{range}(\widehat{S}_c)$ . Most of the discussion here concerning this issue will be very similar to that of section 2.2 on one-level FETI methods. As in the one-level FETI method, we introduce a matrix  $R_c$  such that  $\text{range}(R_c) = \ker(\widehat{S}_c)$ :

$$R_c = \begin{bmatrix} \widehat{R}^{(1)} & & \\ & \ddots & \\ & & \widehat{R}^{(N)} \end{bmatrix},$$

where  $\widehat{R}^{(i)}$  consists of the null vectors of  $\widehat{S}_\Gamma^{(i)}$ ,  $i = 1, \dots, N$ . In the PCR iterations, we will use an initial vector of Lagrange multipliers  $\lambda_0$  which satisfies  $\widehat{g}_c - (\widehat{Z}^k \widehat{B}_{\Gamma,c})^T \lambda_0 \in \text{range}(\widehat{S}_c)$ , and an increment  $\mu$  with  $\widehat{Z}^k \widehat{B}_{\Gamma,c}^T \mu \in \text{range}(\widehat{S}_c)$ . Therefore the space of admissible increments is defined as follows:

$$V^k := \{\mu \in \text{range}(\widehat{B}_{\Gamma,c}) : (\widehat{Z}^k \widehat{B}_{\Gamma,c})^T \mu \in \text{range}(\widehat{S}_c)\} = \ker(G^{kT}),$$

where  $G^k := \widehat{Z}^k \widehat{B}_{\Gamma,c} R_c$ .

We introduce a projection operator  $P^k$  for the Lagrange multipliers which is an orthogonal projection from  $U$  to  $V^k = \ker(G^{kT})$ :

$$P^k := I - G^k (G^{kT} G^k)^{-1} G^{kT}.$$

We also introduce a subspace  $\widehat{W}_{\Gamma,R} := \text{range}(\widehat{S}_c)$  of  $\widehat{W}_{\Gamma,c}$ . We rewrite (4.2) in terms of vectors in the subspace  $\widehat{W}_{\Gamma,R} \times V$ . First, noting that any admissible  $\lambda$  has a decomposition of the form  $\lambda = \lambda_0 + \mu$ ,  $\mu \in V$ , we rewrite the leading equation of (4.2) as

$$\widehat{S}_c u_\Gamma + (\widehat{Z}_\Gamma^k \widehat{B}_{\Gamma,c})^T \mu = \widehat{g}_c - (\widehat{Z}_\Gamma^k \widehat{B}_{\Gamma,c})^T \lambda_0. \quad (4.3)$$

Using (4.3) and that  $P^{kT} \mu = P^k \mu = \mu$ , we can rewrite (4.2):

$$\begin{bmatrix} \widehat{S}_c & (P^k \widehat{Z}_\Gamma^k \widehat{B}_{\Gamma,c})^T \\ \widehat{Z}_\Gamma^k \widehat{B}_{\Gamma,c} & 0 \end{bmatrix} \begin{bmatrix} u_\Gamma \\ \mu \end{bmatrix} = \begin{bmatrix} \widehat{g}_c - \widehat{B}_{\Gamma,c}^T \lambda_0 \\ 0 \end{bmatrix}. \quad (4.4)$$

The solution of (4.4) satisfies

$$\begin{bmatrix} \widehat{S}_c & (P^k \widehat{Z}_\Gamma^k \widehat{B}_{\Gamma,c})^T \\ P^k \widehat{Z}_\Gamma^k \widehat{B}_{\Gamma,c} & 0 \end{bmatrix} \begin{bmatrix} u_\Gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} \widehat{g}_c - \widehat{B}_{\Gamma,c}^T \lambda_0 \\ 0 \end{bmatrix}. \quad (4.5)$$

We use the system (4.5) in order to make sure that our iterates are in the subspace  $\widehat{W}_{\Gamma,R} \times V$ . Note that the displacement part of the solution of (4.5),  $u_\Gamma$ , does not necessarily satisfy the continuity condition  $\widehat{Z}^k \widehat{B}_{\Gamma,c} u_\Gamma = 0$ ; we can recover a solution which satisfies all constraints via the operation  $u_\Gamma - R_c (G^{kT} G^k)^{-1} G^{kT} \widehat{B}_{\Gamma,c} u_\Gamma$ , see [15, 20, 18].

We now consider a second approach to solve (3.5), following [9]; (3.5) can be rewritten as

$$\min_{u_\Gamma \in \widehat{W}_{\Gamma,c}} \frac{1}{2} u_\Gamma^T \widehat{S}_c u_\Gamma - \widehat{g}_c^T u_\Gamma, \quad \text{with } \widehat{B}_{\Gamma,c} u_\Gamma \leq 0, \quad (4.6)$$

which is equivalent to the following problem

$$\begin{cases} \widehat{S}_c u_\Gamma + \widehat{B}_{\Gamma,c}^T \lambda = \widehat{g}_c \\ \widehat{B}_{\Gamma,c} u_\Gamma \leq 0, \quad \lambda \geq 0, \quad \lambda^T \widehat{B}_{\Gamma,c} u_\Gamma = 0. \end{cases} \quad (4.7)$$

The complementarity condition given in the second line is equivalent to

$$C(u_\Gamma, \lambda) := \lambda - \max(0, \lambda + c \widehat{B}_{\Gamma,c} u_\Gamma) = 0, \quad (4.8)$$

for each  $c > 0$ . The system (4.7) can thus be expressed as the following nonlinear system of equations:

$$\begin{cases} \widehat{S}_c u_\Gamma + \widehat{B}_{\Gamma,c}^T \lambda = \widehat{g}_c \\ C(u_\Gamma, \lambda) = 0. \end{cases} \quad (4.9)$$

It follows that a Newton step for the nonlinear system (4.9) is

$$\begin{bmatrix} \widehat{S}_c & \widehat{B}_{\Gamma,c}^T \\ c \widehat{B}_{\Gamma,c, \mathcal{A}^k} & -I_{\mathcal{I}^k} \end{bmatrix} \begin{bmatrix} \delta u_\Gamma^k \\ \delta \lambda^k \end{bmatrix} = \begin{bmatrix} \widehat{g}_c - (\widehat{S}_c u_\Gamma^k + \widehat{B}_{\Gamma,c}^T \lambda^k) \\ C(u^k, \lambda^k) \end{bmatrix} \quad (4.10)$$

and

$$u_\Gamma^{k+1} = u_\Gamma^k + \delta u_\Gamma^k, \quad \lambda^{k+1} = \lambda^k + \delta \lambda^k. \quad (4.11)$$

where

$$\mathcal{I}_k = \{i : (\lambda^k + c \widehat{B}_{\Gamma,c} u_\Gamma^k)_i \leq 0\}, \quad \mathcal{A}_k = \{i : (\lambda^k + c \widehat{B}_{\Gamma,c} u_\Gamma^k)_i > 0\}, \quad (4.12)$$

and  $\widehat{B}_{\Gamma,c, \mathcal{A}^k}$  results from replacing row  $i$  of  $\widehat{B}_{\Gamma,c}$  with zeros, for all  $i \in \mathcal{A}^k$ .  $I_{\mathcal{I}^k}$  is defined similarly.

We can rewrite the second equation of (4.10) as follows:

$$(c \widehat{B}_{\Gamma,c} \delta u_\Gamma^k)_i = -(c \widehat{B}_{\Gamma,c} u_\Gamma^k)_i, \quad \forall i \in \mathcal{A}_k, \quad \text{and} \quad -(\delta \lambda^k)_i = \lambda_i^k, \quad i \in \mathcal{I}_k, \quad (4.13)$$

and the first equation as

$$\widehat{S}_c \delta u_\Gamma^k + \widehat{B}_{\Gamma,c, \mathcal{A}^k}^T (\delta \lambda^k)_{\mathcal{A}^k} + \widehat{B}_{\Gamma,c, \mathcal{I}^k}^T (\delta \lambda^k)_{\mathcal{I}^k} = \widehat{g}_c - (\widehat{S}_c u_\Gamma^k + \widehat{B}_{\Gamma,c, \mathcal{A}^k}^T (\lambda^k)_{\mathcal{A}^k} + \widehat{B}_{\Gamma,c, \mathcal{I}^k}^T (\lambda^k)_{\mathcal{I}^k}), \quad (4.14)$$

which is equivalent to

$$\widehat{S}_c \delta u_\Gamma^k + \widehat{B}_{\Gamma,c, \mathcal{A}^k}^T (\delta \lambda^k)_{\mathcal{A}^k} = \widehat{g}_c - (\widehat{S}_c u_\Gamma^k + \widehat{B}_{\Gamma,c, \mathcal{A}^k}^T (\lambda^k)_{\mathcal{A}^k}), \quad (4.15)$$

due to (4.13).

Consequently, we can rewrite the Newton step defined by (4.10) and (4.11) as

$$\begin{bmatrix} \widehat{S}_c & \widehat{B}_{\Gamma,c, \mathcal{A}^k}^T \\ c \widehat{B}_{\Gamma,c, \mathcal{A}^k} & 0 \end{bmatrix} \begin{bmatrix} \delta u_\Gamma^k \\ \delta \lambda^k \end{bmatrix} = \begin{bmatrix} \widehat{g}_c - (\widehat{S}_c u_\Gamma^k + \widehat{B}_{\Gamma,c, \mathcal{A}^k}^T (\lambda^k)_{\mathcal{A}^k}) \\ -c \widehat{B}_{\Gamma,c, \mathcal{A}^k} u_\Gamma^k \end{bmatrix} \quad (4.16)$$

and

$$u_\Gamma^{k+1} = u_\Gamma^k + \delta u_\Gamma^k, \quad \lambda^{k+1} = \lambda^k + \delta \lambda^k, \quad \text{where} \quad -(\delta \lambda^k)_i = \lambda_i^k, \quad i \in \mathcal{I}_k. \quad (4.17)$$

Note that with a given pair of  $(u_\Gamma^k, \lambda^k)$ , (4.16) and (4.17) are equivalent to

$$\begin{bmatrix} \widehat{S}_c & \widehat{B}_{\Gamma,c, \mathcal{A}^k}^T \\ c \widehat{B}_{\Gamma,c, \mathcal{A}^k} & 0 \end{bmatrix} \begin{bmatrix} u_\Gamma^{k+1} \\ \lambda^{k+1} \end{bmatrix} = \begin{bmatrix} \widehat{g}_c \\ 0 \end{bmatrix}. \quad (4.18)$$

Notice the similarity between (4.2) and (4.18); in fact,  $\widehat{B}_{\Gamma,c, \mathcal{A}^k} = \widehat{Z}^k \widehat{B}_{\Gamma,c}$  and with  $c = 1$ , (4.2) and (4.18) are the same. In the following we will solve (4.9), with  $c = 1$ , via Newton's method as defined by (4.16) and (4.17).

We now discuss our choice of preconditioner. Let  $A_{OL}^{(i)}$  denote the stiffness matrix for the entire body

$\Omega_i$ : this needs to be distinguished from  $A^{(i)}$ , which is a direct sum of stiffness matrices for individual subdomains (see (3.4)). We have

$$A_{OL}^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{\Gamma I}^{(i)T} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix}, \quad i = 1, \dots, N,$$

where  $A_{\Gamma\Gamma}^{(i)}$  corresponds to the nodes on  $\partial\Omega_i \cap \overline{\Gamma_{gl}}$  and  $A_{II}^{(i)}$  to those interior of  $\Omega_i$ , etc. We define the corresponding Schur complement  $S_{OL}^{(i)} := A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)-1} A_{\Gamma I}^{(i)T}$  and also a block-diagonal matrix for the entire system as the direct sum of the Schur complements for the individual bodies:

$$S_{OL} := \begin{bmatrix} S_{OL}^{(1)} & & \\ & \ddots & \\ & & S_{OL}^{(N)} \end{bmatrix}.$$

Since inverting  $A_{II}^{(i)}$  can be expensive in practice, we need to solve  $A_{II}^{(i)}x = b$  approximately; we propose a way of doing this in Section 7.

We now introduce the following block-diagonal preconditioner for the system:

$$\mathcal{B}^{-1} = \begin{bmatrix} P_R M_{BDDC}^{-1} P_R & 0 \\ 0 & P^k M_D^{-1} P^k \end{bmatrix} \quad (4.19)$$

where  $P_R := I - R_c(R_c^T R_c)^{-1} R_c^T$  is an orthogonal projection operator onto  $\text{range}(\widehat{S}_c)$  and

$$M_{BDDC}^{-1} = \begin{bmatrix} \bar{R}_{D,\Gamma}^{(1)T} \widetilde{S}_\Gamma^{(1)\dagger} \bar{R}_{D,\Gamma}^{(1)} & & \\ & \ddots & \\ & & \bar{R}_{D,\Gamma}^{(N)T} \widetilde{S}_\Gamma^{(N)\dagger} \bar{R}_{D,\Gamma}^{(N)} \end{bmatrix},$$

$$M_D^{-1} = \widehat{Z}_\Gamma^k B_{\Gamma_{OL},D} S_{OL} B_{\Gamma_{OL},D}^T \widehat{Z}_\Gamma^{kT}.$$

We rewrite the KKT system (4.5) as

$$\mathcal{A}x = \mathcal{F}, \quad (4.20)$$

where

$$\mathcal{A} := \begin{bmatrix} \widehat{S}_c & (P^k \widehat{Z}_\Gamma^k \widehat{B}_{\Gamma,c})^T \\ P^k \widehat{Z}_\Gamma^k \widehat{B}_{\Gamma,c} & 0 \end{bmatrix}, \quad x := \begin{bmatrix} u_\Gamma \\ \lambda \end{bmatrix} \quad \text{and} \quad \mathcal{F} := \begin{bmatrix} \widehat{g}_c \\ 0 \end{bmatrix}. \quad (4.21)$$

In our hybrid method, we use a preconditioned conjugate residual (PCR) method with  $\mathcal{B}$  as the preconditioner. For a description of the preconditioned conjugate residual method, see [26, 8].

## 5 Convergence Estimates

Suppose the following system is solved with the PCR method with the preconditioner  $M$ :

$$Au = b. \quad (5.1)$$

We define

$$\mathcal{K}(M^{-1}A) = \frac{\mu_{max}}{\mu_{min}} = \frac{\max\{|\lambda| : \lambda \in \sigma(M^{-1}A)\}}{\min\{|\lambda| : \lambda \in \sigma(M^{-1}A)\}}, \quad (5.2)$$

where  $\sigma(M^{-1}A)$  is the spectrum of  $M^{-1}A$ . We have the following result, see [26, C.6.2]; a proof can be found in [8, Section 9.5].

**Lemma 5.1.** *Let  $A$  be regular and symmetric and  $M$  symmetric and positive definite. Then, after  $k$  steps of the PCR algorithm, the norm of the residual is bounded by*

$$\|M^{-1/2}r^k\|_2 \leq \frac{2\rho^\mu}{1+\rho^{2\mu}}\|M^{-1/2}r^0\|_2,$$

where  $\rho = \frac{\kappa-1}{\kappa+1}$  and  $\mu \in \mathbb{Z}$ , such that  $k/2 - 1 < \mu \leq k/2$ .

According to Lemma 5.1, we need to study the spectrum of the preconditioned operator  $\mathcal{B}^{-1}\mathcal{A}$ , which has the same spectrum as  $\mathcal{B}^{-1/2}\mathcal{A}\mathcal{B}^{-1/2}$ .

**General Case** We first study a general case, where

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}, \quad \mathcal{B}^{-1} = \begin{bmatrix} \hat{A}^{-1} & 0 \\ 0 & \hat{C}^{-1} \end{bmatrix},$$

assuming that

$$\alpha_0 u^T \hat{A} u \leq u^T A u \leq \alpha_1 u^T \hat{A} u, \quad \forall u. \quad (5.3)$$

We assume  $A, \hat{A} \in \mathbb{R}^{n \times n}, \hat{C} \in \mathbb{R}^{m \times m}$  are real symmetric and positive definite. Then,

$$\mathcal{B}^{-1/2}\mathcal{A}\mathcal{B}^{-1/2} = \begin{bmatrix} \hat{A}^{-1/2}A\hat{A}^{-1/2} & \hat{A}^{-1/2}B^T\hat{C} \\ \hat{C}^{-1/2}B\hat{A}^{-1/2} & 0 \end{bmatrix}.$$

In the following, we use the notation  $\tilde{A} := \hat{A}^{-1/2}A\hat{A}^{-1/2}$  and  $\tilde{B} := \hat{C}^{-1/2}B\hat{A}^{-1/2}$ . Note that

$$\alpha_0 u^T u \leq u^T \tilde{A} u \leq \alpha_1 u^T u, \quad \forall u. \quad (5.4)$$

We study the cases where  $\hat{A} = A$  and  $\hat{A} \neq A$  separately. When  $\hat{A} = A$ ,  $\tilde{A}$  is simply the identity matrix and

$$\mathcal{B}^{-1/2}\mathcal{A}\mathcal{B}^{-1/2} = \begin{bmatrix} I & \tilde{B}^T \\ \tilde{B} & 0 \end{bmatrix}. \quad (5.5)$$

**Lemma 5.2.** *Let  $\mathcal{B}^{-1/2}\mathcal{A}\mathcal{B}^{-1/2}$  be defined as in (5.5). We then have*

$$\mathcal{K}(\mathcal{B}^{-1}\mathcal{A}) = \mathcal{K}(\mathcal{B}^{-1/2}\mathcal{A}\mathcal{B}^{-1/2}) = \frac{1/2 + \sqrt{1/4 + \lambda_{max}}}{-1/2 + \sqrt{1/4 + \lambda_{min}}},$$

where  $\lambda_{max}$  and  $\lambda_{min}$  are the largest and smallest eigenvalues of  $\tilde{B}^T\tilde{B}$ , respectively.

*Proof.* We consider the following eigenvalue problem:

$$\begin{bmatrix} I & \tilde{B}^T \\ \tilde{B} & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = t \begin{bmatrix} u \\ \lambda \end{bmatrix},$$

which is equivalent to

$$\begin{aligned} u + \tilde{B}^T \lambda &= tu \\ \tilde{B} u &= t\lambda \end{aligned} \quad (5.6)$$

Notice that  $t \neq 0$  due to the nonsingularity of  $\mathcal{A}$  and  $\mathcal{B}$ . Substituting the second equation of (5.6) into the first, we obtain

$$u + t^{-1}\tilde{B}^T\tilde{B}u = tu.$$

Denoting the eigenvalues of  $\tilde{B}^T\tilde{B}$  by  $\lambda_i \geq 0, i = 1, \dots, n$ , we obtain

$$(1 + \lambda_i/t - t)u = 0, \quad i = 1, \dots, n.$$

Since  $u = 0$  leads to  $\lambda = 0$ , we need to solve  $1 + \lambda_i/t - t = 0, i = 1, \dots, n$ , which are equivalent to the quadratic equations  $t^2 - t - \lambda_i = 0$ . Their solutions are  $1/2 \pm \sqrt{1/4 + \lambda_i}$  and thus

$$\sigma(\mathcal{B}^{-1}\mathcal{A}) = \{1/2 \pm \sqrt{1/4 + \lambda_i} : i = 1, \dots, n\}.$$

Clearly,

$$\begin{aligned} \max\{|\lambda| : \lambda \in \sigma(\mathcal{B}^{-1}\mathcal{A})\} &= 1/2 + \sqrt{1/4 + \lambda_{max}} \text{ and} \\ \min\{|\lambda| : \lambda \in \sigma(\mathcal{B}^{-1}\mathcal{A})\} &= -1/2 + \sqrt{1/4 + \lambda_{min}}, \end{aligned}$$

where  $\lambda_{max} := \max_{i=1}^n \lambda_i$  and  $\lambda_{min} := \min_{i=1}^n \lambda_i$ . □

We now consider the case  $\hat{A} \neq A$ . Then the eigenvalue analysis of  $\mathcal{A}_1 := \mathcal{B}^{-1/2}\mathcal{A}\mathcal{B}^{-1/2} = \begin{bmatrix} \tilde{A} & \tilde{B}^T \\ \tilde{B} & 0 \end{bmatrix}$  is not as easy, and we left- and right- multiply this symmetrized preconditioned operator with  $\mathcal{C}^{-1/2} = \begin{bmatrix} \tilde{A}^{-1/2} & 0 \\ 0 & I \end{bmatrix}$  to obtain

$$\mathcal{A}_2 := \mathcal{C}^{-1/2}\mathcal{A}_1\mathcal{C}^{-1/2} = \begin{bmatrix} I & \tilde{A}^{-1/2}\tilde{B}^T \\ \tilde{B}\tilde{A}^{-1/2} & 0 \end{bmatrix}. \quad (5.7)$$

Eigenvalues of  $\mathcal{A}_2$  can be analyzed in the same manner as in Lemma 5.2. To relate the spectrum of  $\mathcal{A}_1$  to the spectrum of  $\mathcal{A}_2$ , we use the Courant-Fischer Minimax Theorem.

**Theorem 5.3** (Courant-Fischer). *Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix with real eigenvalues  $\lambda_i, i = 1, \dots, n$ , which are ordered so that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Then*

$$\lambda_k = \max_{\dim(V)=k} \min_{\substack{x \in V \\ x \neq 0}} \frac{x^T A x}{x^T x} \quad (5.8)$$

$$\lambda_k = \min_{\dim(V)=n-k+1} \max_{\substack{x \in V \\ x \neq 0}} \frac{x^T A x}{x^T x} \quad (5.9)$$

Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  denote the eigenvalues of  $\mathcal{A}_2$ , and  $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \dots \geq \tilde{\lambda}_n$  the eigenvalues of  $\mathcal{A}_1$ . Suppose  $\lambda_k > 0$  and  $\lambda_{k+1} < 0$ , where  $\lambda_k$  is the smallest positive eigenvalue of  $\mathcal{A}_2$  and  $\lambda_{k+1}$  the largest negative eigenvalue of  $\mathcal{A}_2$ . Also, let  $q_i, i = 1, \dots, n$  denote the eigenvectors of  $\mathcal{A}_2$  such that  $\mathcal{A}_2 q_i = \lambda_i q_i$  and  $q_i^T q_j = \delta_{ij}, i, j = 1, \dots, n$ . Using (5.8) and the fact that  $\mathcal{A}_1 = \mathcal{C}^{1/2}\mathcal{A}_2\mathcal{C}^{1/2}$  we have

$$\tilde{\lambda}_k = \max_{\dim(V)=k} \min_{\substack{x \in V \\ x \neq 0}} \frac{x^T \mathcal{A}_1 x}{x^T x} = \max_{\dim(V)=k} \min_{\substack{x \in V \\ x \neq 0}} \frac{(\mathcal{C}^{1/2}x)^T \mathcal{A}_2 (\mathcal{C}^{1/2}x)}{(\mathcal{C}^{1/2}x)^T (\mathcal{C}^{1/2}x)} \frac{(\mathcal{C}^{1/2}x)^T (\mathcal{C}^{1/2}x)}{x^T x}$$

For  $V := \mathcal{C}^{-1/2}\text{span}\{q^{(1)}, q^{(2)}, \dots, q^{(k)}\}$ , we have

$$\min_{\substack{x \in V \\ x \neq 0}} \frac{(\mathcal{C}^{1/2}x)^T \mathcal{A}_2 (\mathcal{C}^{1/2}x)}{(\mathcal{C}^{1/2}x)^T (\mathcal{C}^{1/2}x)} \geq \lambda_k.$$

Noting that

$$\alpha_0 x^T x \leq x^T \mathcal{C} x \leq \alpha_1 x^T x, \quad \forall x \quad (5.10)$$

due to the definition of  $\mathcal{C}$  and (5.4), we have

$$\min_{\substack{x \in V \\ x \neq 0}} \frac{(\mathcal{C}^{1/2}x)^T \mathcal{A}_1 (\mathcal{C}^{1/2}x)}{(\mathcal{C}^{1/2}x)^T (\mathcal{C}^{1/2}x)} \frac{(\mathcal{C}^{1/2}x)^T (\mathcal{C}^{1/2}x)}{x^T x} \geq \lambda_k \alpha_0.$$

Taking the maximum over all  $k$ -dimensional subspaces on the left hand side of the previous equation, we obtain

$$\tilde{\lambda}_k \geq \lambda_k \alpha_0.$$

Similarly, using (5.9), we have

$$\tilde{\lambda}_{k+1} = \min_{\dim(V)=n-k} \max_{\substack{x \in V \\ x \neq 0}} \frac{x^T \mathcal{A}_1 x}{x^T x} = \min_{\dim(V)=n-k} \max_{\substack{x \in V \\ x \neq 0}} \frac{(\mathcal{C}^{1/2} x)^T \mathcal{A}_2 (\mathcal{C}^{1/2} x)}{(\mathcal{C}^{1/2} x)^T (\mathcal{C}^{1/2} x)} \frac{(\mathcal{C}^{1/2} x)^T (\mathcal{C}^{1/2} x)}{x^T x}$$

For  $V := \mathcal{C}^{-1/2}\{q^{(k+1)}, \dots, q^{(n)}\}$ , we have

$$\max_{\substack{x \in V \\ x \neq 0}} \frac{(\mathcal{C}^{1/2} x)^T \mathcal{A}_2 (\mathcal{C}^{1/2} x)}{(\mathcal{C}^{1/2} x)^T (\mathcal{C}^{1/2} x)} \leq \lambda_{k+1}$$

and

$$\max_{\substack{x \in V \\ x \neq 0}} \frac{(\mathcal{C}^{1/2} x)^T \mathcal{A}_2 (\mathcal{C}^{1/2} x)}{(\mathcal{C}^{1/2} x)^T (\mathcal{C}^{1/2} x)} \frac{(\mathcal{C}^{1/2} x)^T (\mathcal{C}^{1/2} x)}{x^T x} \leq \lambda_{k+1} \alpha_1.$$

Taking the minimum on the left hand side of the previous equation, we obtain

$$\tilde{\lambda}_{k+1} \leq \lambda_{k+1} \alpha_1.$$

By a similar argument,

$$\tilde{\lambda}_1 \leq \lambda_1 \alpha_1 \quad \text{and} \quad \tilde{\lambda}_n \geq \lambda_n \alpha_0.$$

Letting  $\lambda'_{max}$  and  $\lambda'_{min}$  denote the maximum and the minimum eigenvalues of  $\tilde{A}^{-1/2} \tilde{B}^T \tilde{B} \tilde{A}^{-1/2}$ , respectively, we obtain

$$\begin{aligned} \mathcal{K}(\mathcal{A}_1) &= \frac{\max\{\tilde{\lambda}_1, |\tilde{\lambda}_n|\}}{\min\{\tilde{\lambda}_k, |\tilde{\lambda}_{k+1}|\}} \leq \frac{\alpha_1 \max\{\lambda_1, |\lambda_n|\}}{\alpha_0 \min\{\lambda_k, |\lambda_{k+1}|\}} \\ &= \frac{\alpha_1}{\alpha_0} \mathcal{K}(\mathcal{A}_2) \leq \frac{\alpha_1}{\alpha_0} \frac{1/2 + \sqrt{1/4 + \lambda'_{max}}}{-1/2 + \sqrt{1/4 + \lambda'_{min}}}, \end{aligned} \quad (5.11)$$

where the second inequality follows from the definition of  $\mathcal{A}_2$  in (5.7) and Lemma 5.2. Noticing that

$$\begin{aligned} \lambda_{max}(\tilde{A}^{-1/2} \tilde{B}^T \tilde{B} \tilde{A}^{-1/2}) &\leq \lambda_{max}(\tilde{B}^T \tilde{B}) \lambda_{max}(\tilde{A}^{-1}) \\ \lambda_{min}(\tilde{A}^{-1/2} \tilde{B}^T \tilde{B} \tilde{A}^{-1/2}) &\geq \lambda_{min}(\tilde{B}^T \tilde{B}) \lambda_{min}(\tilde{A}^{-1}), \end{aligned}$$

and

$$\frac{1}{\alpha_1} u^T u \leq u^T \tilde{A}^{-1} u \leq \frac{1}{\alpha_0} u^T u, \quad \forall u,$$

we rewrite (5.11) in terms of  $\lambda_{max}$  and  $\lambda_{min}$ , the maximum and the minimum eigenvalues of  $\tilde{B}^T \tilde{B}$  and obtain:

$$\mathcal{K}(\mathcal{A}_1) \leq \frac{\alpha_1}{\alpha_0} \frac{1/2 + \sqrt{1/4 + \lambda_{max}/\alpha_0}}{-1/2 + \sqrt{1/4 + \lambda_{min}/\alpha_1}}. \quad (5.12)$$

**Special Case** We now use these results to study the convergence bound of our preconditioned system  $\mathcal{B}^{-1} \mathcal{A}$ , where  $\mathcal{B}^{-1}$  and  $\mathcal{A}$  are defined in (4.19) and (4.21), respectively. We have

$$A = \hat{S}_\Gamma, \quad B = P^k \hat{Z}_\Gamma^k \hat{B}_{\Gamma,c}, \quad \hat{A}^{-1} = P_R M_{BDDC}^{-1} P_R, \quad \hat{C}^{-1} = P^k M_D^{-1} P^k.$$

Notice that  $A, \hat{A}$ , and  $\hat{C}$  are now singular. However, this does not pose any problem, since in the application of the PCR method our iterates will be in a proper subspace in which those matrices will be nonsingular. From (5.12), we can see that the extreme eigenvalues of  $\tilde{B}^T \tilde{B}$  and  $\alpha_0, \alpha_1$  in (5.3) are important parameters, where  $\tilde{B}^T \tilde{B} = \hat{A}^{-1/2} B^T \hat{C}^{-1} B \hat{A}^{-1/2}$ , which has the same spectrum as  $B \hat{A}^{-1} B^T \hat{C}^{-1}$ . In our case,

$$\begin{aligned} &B \hat{A}^{-1} B^T \hat{C}^{-1} \\ &= P^k \hat{Z}_\Gamma^k \hat{B}_{\Gamma,c} P_R M_{BDDC}^{-1} P_R \hat{B}_{\Gamma,c}^T \hat{Z}_\Gamma^k P^k \cdot P^k M_D^{-1} P^k. \end{aligned} \quad (5.13)$$

The following lemma indicates the spectral equivalence between the matrices  $A$  and  $\hat{A}$ ; for a proof, see [21].

**Lemma 5.4.**

$$x^T \widehat{A}x \leq x^T Ax \leq C \left(1 + \log \left(\frac{H_s}{h}\right)\right)^2 x^T \widehat{A}x, \quad (5.14)$$

for all  $x \in \text{range}(\widehat{S}_\Gamma)$ .

Thus, we can study the spectrum of

$$\begin{aligned} & BA^\dagger B^T \widehat{C}^{-1} \\ &= P^k \widehat{Z}_\Gamma^k \widehat{B}_{\Gamma,c} P_R \widehat{S}_\Gamma^\dagger P_R \widehat{B}_{\Gamma,c}^T \widehat{Z}_\Gamma^{k^T} P^k \cdot P^k M_D^{-1} P^k \\ &= P^k \widehat{Z}_\Gamma^k \widehat{B}_{\Gamma,c} \widehat{S}_\Gamma^\dagger \widehat{B}_{\Gamma,c}^T \widehat{Z}_\Gamma^{k^T} P^k \cdot P^k \widehat{Z}_\Gamma^k B_{\Gamma_{OL},D} S_{OL} B_{\Gamma_{OL},D}^T \widehat{Z}_\Gamma^{k^T} P^k \end{aligned} \quad (5.15)$$

instead of that of the matrix (5.13), where

$$\widehat{S}_\Gamma^\dagger = \begin{bmatrix} \widehat{S}_\Gamma^{(1)\dagger} & & \\ & \ddots & \\ & & \widehat{S}_\Gamma^{(N)\dagger} \end{bmatrix}.$$

**Lemma 5.5.**

$$\widehat{B}_{\Gamma,c} \widehat{S}_\Gamma^\dagger \widehat{B}_{\Gamma,c}^T \widehat{Z}_\Gamma^{k^T} P^k = B_{\Gamma_{OL}} S_{OL}^\dagger B_{\Gamma_{OL}}^T \widehat{Z}_\Gamma^{k^T} P^k.$$

*Proof.* Let  $\widehat{B}_{\Gamma,c} = [\widehat{B}_\Gamma^{(1)}, \dots, \widehat{B}_\Gamma^{(N)}]$ ,  $B_{\Gamma_{OL}} = [B_{\Gamma_{OL}}^{(1)}, \dots, B_{\Gamma_{OL}}^{(N)}]$ . Note that the solution of  $\widehat{S}_\Gamma^{(i)} u_\Gamma^{(i)} = \widehat{B}_\Gamma^{(i)T} v$ , where  $\widehat{B}_\Gamma^{(i)T} v \in \text{range}(\widehat{S}_\Gamma^{(i)})$ , can be obtained from the following equation:

$$\begin{aligned} & \begin{bmatrix} A_{II}^{(i,1)} & & & & A_{\Gamma I}^{(i,1)T} R_\Gamma^{(i,1)} \\ & A_{II}^{(i,2)} & & & A_{\Gamma I}^{(i,2)T} R_\Gamma^{(i,2)} \\ & & \ddots & & \vdots \\ & & & A_{II}^{(i,N_i)} & A_{\Gamma I}^{(i,N_i)T} R_\Gamma^{(i,N_i)} \\ R_\Gamma^{(i,1)T} A_{\Gamma I}^{(i,1)} & R_\Gamma^{(i,2)T} A_{\Gamma I}^{(i,2)} & \dots & R_\Gamma^{(i,N_i)T} A_{\Gamma I}^{(i,N_i)} & \sum_{j=1}^{N_i} R_\Gamma^{(i,j)T} A_{\Gamma I}^{(i,j)} R_\Gamma^{(i,j)} \end{bmatrix} \begin{bmatrix} u_I^{(i,1)} \\ u_I^{(i,2)} \\ \vdots \\ u_I^{(i,N_i)} \\ \widehat{u}_\Gamma^{(i)} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \widehat{B}_\Gamma^{(i)T} v \end{bmatrix}, \end{aligned} \quad (5.16)$$

where  $R_\Gamma^{(i,j)} : \widehat{W}_\Gamma^{(i)} \rightarrow W_\Gamma^{(i,j)}$  is a restriction operator. Noting that all entries of  $\widehat{B}_\Gamma^{(i)T} v$ , corresponding to the nodes on  $\Gamma_{loc}^{(i)}$ , are zero and eliminating those entries results in  $B_{\Gamma_{OL}}^{(i)T} v$ , we can rearrange the system (5.16):

$$A_{OL}^{(i)} u^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{\Gamma I}^{(i)T} \\ A_{\Gamma I}^{(i)} & A_{\Gamma \Gamma}^{(i)} \end{bmatrix} \begin{bmatrix} u_\Gamma^{(i)} \\ u_\Gamma^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ B_{\Gamma_{OL}}^{(i)T} v \end{bmatrix} \quad (5.17)$$

where  $u_\Gamma^{(i)}$  is the displacement on  $\overline{\Gamma_{gl}} \cap \partial\Omega_i$ . The equivalence of (5.16) and (5.17) shows that  $\widehat{B}_\Gamma^{(i)} \widehat{S}_\Gamma^{(i)\dagger} \widehat{B}_\Gamma^{(i)T} \widehat{Z}_\Gamma^{k^T} P^k = B_{\Gamma_{OL}}^{(i)} S_{OL}^{(i)\dagger} B_{\Gamma_{OL}}^{(i)T} \widehat{Z}_\Gamma^{k^T} P^k$ .  $\square$

Due to Lemma 5.5, the operator (5.15) can be written as

$$P^k \widehat{Z}_\Gamma^k B_{\Gamma_{OL}} S_{OL}^\dagger B_{\Gamma_{OL}}^T \widehat{Z}_\Gamma^{k^T} P^k \cdot P^k \widehat{Z}_\Gamma^k B_{\Gamma_{OL},D} S_{OL} B_{\Gamma_{OL},D}^T \widehat{Z}_\Gamma^{k^T} P^k. \quad (5.18)$$

The proof of the following lemma proceeds, line by line, as the proof of [26, Theorem 6.15].

**Lemma 5.6.** For  $\forall \lambda \in \text{range}(P^k)$ ,

$$\begin{aligned} & \langle \lambda, \lambda \rangle \\ & \leq \langle P^k \widehat{Z}_\Gamma^k B_{\Gamma_{OL}} S_{OL}^\dagger B_{\Gamma_{OL}}^T \widehat{Z}_\Gamma^{kT} P^k \cdot P^k \widehat{Z}_\Gamma^k B_{\Gamma_{OL,D}} S_{OL} B_{\Gamma_{OL,D}}^T \widehat{Z}_\Gamma^{kT} P^k \lambda, \lambda \rangle \\ & \leq C(1 + \log(H_b/h))^2 \langle \lambda, \lambda \rangle. \end{aligned}$$

We now can derive a concrete bound for (5.12), using Lemmas 5.4 and 5.6. In our case,  $\lambda_{max} = C(1 + \log(H_b/h))^2(1 + \log(H_s/h))^2$ ,  $\lambda_{min} = 1$ ,  $\alpha_1 = C(1 + \log(H_s/h))^2$ , and  $\alpha_0 = 1$ . Assuming that  $H_b/h$  and  $H_s/h$  are large enough, we have

$$\frac{1}{2} + \sqrt{\frac{1}{2} + \frac{\lambda_{max}}{\alpha_0}} \approx \sqrt{\frac{\lambda_{max}}{\alpha_0}}, \quad (5.19)$$

and

$$-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\lambda_{min}}{\alpha_1}} = -\frac{1}{2} + \frac{1}{2} \sqrt{1 + 4 \frac{\lambda_{min}}{\alpha_1}} = -\frac{1}{2} + \frac{1}{2} \left( 1 + 2 \frac{\lambda_{min}}{\alpha_1} + O\left(\left(4 \frac{\lambda_{min}}{\alpha_1}\right)^2\right) \right). \quad (5.20)$$

Combining (5.12), (5.19), and (5.20), we have

$$\mathcal{K}(\mathcal{A}_1) \leq C(1 + \log(H_b/h))(1 + \log(H_s/h))^5. \quad (5.21)$$

We obtain

**Theorem 5.7.** Let  $\mathcal{B}^{-1}$ ,  $\mathcal{A}$ , and  $\mathcal{K}(\mathcal{B}^{-1}\mathcal{A})$  be defined as in (4.19), (4.21), and (5.2), respectively. Then we have the following bound:

$$\mathcal{K}(\mathcal{B}^{-1}\mathcal{A}) \leq C(1 + \log(H_b/h))(1 + \log(H_s/h))^5.$$

## 6 Numerical Experiments

Recall that an active set method consists of outer iterations, in which the active set is updated, and inner iterations, in which auxiliary equality constrained problems are solved on the current active set. In this section, we solve the latter by using the hybrid method. We note that such problems were solved using the FETI-FETI method in [20, 18].

We solve the following minimization problem:

$$\min \sum_{i=1}^{N_b \times N_b} \left( \frac{1}{2} \int_{\Omega_i} |\nabla u^i|^2 dx - \int_{\Omega_i} f u^i dx \right), \quad (6.1)$$

where  $\Omega_i \subset \mathbb{R}^2$ ,  $i = 1, \dots, N_b \times N_b$ , are square bodies with side length  $H_b := 1/N_b$  which form the system  $\Omega = \bigcup_{i=1}^{N_b \times N_b} \Omega_i = [0, 1] \times [0, 1]$ . We require  $u^i \in H^1(\Omega_i)$ ,  $u^i|_{\partial\Omega_i \cap \partial\Omega} = 0$ . Each  $\Omega_i$  is decomposed into  $N_s \times N_s$  square subdomains, each of which is discretized by square bilinear elements of side length  $h$ . Also,  $\Gamma := \bigcup_{i \neq j} \partial\Omega_i \cap \partial\Omega_j$  denotes the interface between the bodies.

We consider two linearized problems, each with a different *contact area* between the bodies. In the first problem, the entire  $\Gamma$  is considered as the contact area, i.e., we require the continuity of the displacement vector across the entire  $\Gamma$ . In the second problem, continuity is imposed only on the middle third of the faces between the bodies. We use the preconditioned conjugate residual method. All our experiments have been performed in MATLAB, and the stopping criterion is  $\|r_n\|_2 / \|r_0\|_2 < 10^{-5}$ , where  $r_n$  and  $r_0$  are the  $n$ th and initial residuals, respectively.

In Table 1, the results obtained with the hybrid method are presented. We have three parameters; the number of bodies across  $\Omega$  ( $N_b = 1/H_b$ ), the number of subdomains across each body ( $N_s = H_b/H_s$ ), and the number of elements across each subdomain ( $H_s/h$ ). We vary one parameter while keeping the

Table 1: Results for the hybrid method. iter denotes the iteration counts. Area on which continuity is imposed between bodies:  $\Gamma$ , i.e., the entire interface for (I), and only a proper subset of  $\Gamma$ ,  $\Gamma_0$  for (II)

			(I)	(II)
$1/H_b$	$H_b/H_s$	$H_s/h$	iter	iter
2	2	2	10	10
4			12	11
6			12	11
8			11	11
10			11	11
12			11	11
2	4	2	10	10
	6		8	10
	8		8	10
	10		8	10
	12		8	9
	14		8	8
	16		7	8
	18		7	7
2	2	4	11	13
		8	13	15
		16	14	16
		32	15	17
		64	16	19
		128	17	20

other two fixed. The results for the first set of experiments, with the entire  $\Gamma$  as the contact surface, are shown in Column (I); those for the second set of experiments with a reduced contact area shown in Column (II). We observe that the iteration counts are independent of  $1/H_b$  and logarithmically dependent on  $H_s/h$ . The iteration counts from Table 1 are also plotted in Figure 3.

Very similar numerical results have been obtained independently by Klawonn and Rheinbach; see [13] and [14].

## 7 Active set method combined with the hybrid method

It is well known that an active set method can often be slow due to a poor initial guess. However, it has been shown in [9] that a certain primal-dual active set strategy, viewed as a Newton method, has a superlinear convergence provided that the initial point is close to the solution. A very efficient strategy of finding a good initial active set and  $\lambda^0$  was discussed briefly in [20, Chapter 5] and in full detail in [19]. In our experiments this turns out to be a good estimate of the optimal active set. We set  $u_{\Gamma}^0 = 0$ . The following is the complete algorithm:

1. Set  $u_{\Gamma}^0 = 0$  and choose  $\lambda^0$  as described in [20, Chapter 5], [19]. Set  $k = 0$ .
2. Set  $\mathcal{I}_k = \{i : (\lambda^k + \widehat{B}_{\Gamma,c} u_{\Gamma}^k)_i \leq 0\}$ ,  $\mathcal{A}_k = \{i : (\lambda^k + \widehat{B}_{\Gamma,c} u_{\Gamma}^k)_i > 0\}$ .
3. Solve

$$\begin{bmatrix} \widehat{S}_c & \widehat{B}_{\Gamma,c,\mathcal{A}^k}^T \\ \widehat{B}_{\Gamma,c,\mathcal{A}^k} & 0 \end{bmatrix} \begin{bmatrix} u_{\Gamma}^{k+1} \\ \lambda^{k+1} \end{bmatrix} = \begin{bmatrix} \widehat{g}_c \\ 0 \end{bmatrix} \quad (7.1)$$

and set  $\lambda^{k+1} = 0$  on  $\mathcal{I}_k$ .



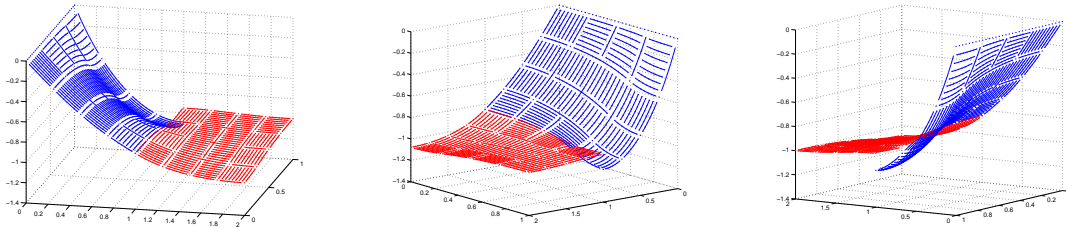
Table 2: Results: primal-dual active set method + hybrid method. *outer it.* denotes the number of outer iterations of the active set method; *inner it.* denotes the number of iterations needed to solve the inner minimization problems by the PCR method, until the norm of the residual has been reduced by  $10^{-5}$ , on the active faces identified in the outer iterations. *total it.* denotes the total number of inner iterations.

$N_{sub}(1/H)$	$H/h$	$N_{dof}(\lambda)$	$N_{dof}(total)$	outer it.	inner it.	total it.
16(4)	4	17	561	2	16 16	32
16(4)	8	33	2145	2	20 19	39
16(4)	12	49	4753	2	22 20	42
16(4)	16	65	8385	2	26 24	50
64(8)	4	33	2145	2	18 17	35
64(8)	8	65	8385	1	23	23
64(8)	12	97	18721	1	27	27
64(8)	16	129	33153	1	29	29
144(12)	4	49	4753	1	19	19
144(12)	8	97	18721	2	24 22	46
144(12)	12	145	41905	2	28 24	52
144(12)	16	193	74305	2	30 27	57
256(16)	4	65	8385	1	19	19
256(16)	8	129	33153	1	26	26
256(16)	12	193	74305	1	28	28
256(16)	16	257	131841	1	32	32

with  $\tilde{R}_{D,\Gamma_0}^{(i)}$  and  $\tilde{S}_{\Gamma_0}^{(i)}$  defined similarly as  $\tilde{R}_{D,\Gamma}^{(i)}$  and  $\tilde{S}_{\Gamma}^{(i)}$ , respectively.

In Table 2, notice that the iteration counts for the inner minimizations does not increase rapidly as we increase the number of elements per subdomain or the number of subdomains per body, which is an indication of the scalability of the hybrid algorithm. Also, notice that it takes at most two outer iterations to reach the optimal solution, which is an indication of the effectiveness of the strategy to find an initial active set [19].

Figure 4: Solution of the model problem, from different angles.  $N_{sub} = 16, H/h = 8$ .



## References

- [1] Philip Avery and Charbel Farhat. The FETI family of domain decomposition methods for inequality-constrained quadratic programming: Application to contact problems with conforming and nonconforming interfaces. *Computer Methods in Applied Mechanics and Engineering*, 198(21-26):1673 – 1683, 2009. Advances in Simulation-Based Engineering Sciences - Honoring J. Tinsley Oden.

- [2] Philip Avery, Gert Rebel, Michel Lesoinne, and Charbel Farhat. A numerically scalable dual-primal substructuring method for the solution of contact problems—part I: the frictionless case. *Comput. Methods Appl. Mech. Engrg.*, 193(23-26):2403–2426, 2004.
- [3] M. Bergounioux, M. Haddou, M. Hintermüller, and K. Kunisch. A comparison of a Moreau–Yosida-based active set strategy and interior point methods for constrained optimal control problems. *SIAM J. on Optimization*, 11(2):495–521, 2000.
- [4] Maïtine Bergounioux, Kazufumi Ito, and Karl Kunisch. Primal-dual strategy for constrained optimal control problems. *SIAM J. Control Optim.*, 37(4):1176–1194, 1999.
- [5] Zdeněk Dostál. *Optimal quadratic programming algorithms. With applications to variational inequalities.*, volume 23 of *Springer Optimization and Its Applications*. Springer, New York, 2009.
- [6] Zdeněk Dostál, David Horák, and Dan Stefanica. A scalable FETI-DP algorithm for a semi-coercive variational inequality. *Comput. Methods Appl. Mech. Engrg.*, 196(8):1369–1379, 2007.
- [7] Charbel Farhat, Po-Shu Chen, and Jan Mandel. A scalable Lagrange multiplier based domain decomposition method for time-dependent problems. *Internat. J. Numer. Methods Engrg.*, 38:3831–3853, 1995.
- [8] Wolfgang Hackbusch. *Iterative solution of large sparse systems of equations*, volume 95 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994. Translated and revised from the 1991 German original.
- [9] M. Hintermüller, K. Ito, and K. Kunisch. The primal-dual active set strategy as a semismooth Newton method. *SIAM J. on Optimization*, 13(3):865–888, 2002.
- [10] Kazufumi Ito and Karl Kunisch. Augmented Lagrangian methods for nonsmooth, convex optimization in Hilbert spaces. *Nonlinear Anal.*, 41(5-6):591–616, 2000.
- [11] Axel Klawonn and Oliver Rheinbach. A parallel implementation of dual-primal FETI methods for three-dimensional linear elasticity using a transformation of basis. *SIAM J. Sci. Comput.*, 28(5):1886–1906, 2006.
- [12] Axel Klawonn and Oliver Rheinbach. Robust FETI-DP methods for heterogeneous three dimensional elasticity problems. *Comput. Methods Appl. Mech. Engrg.*, 196(8):1400–1414, 2007.
- [13] Axel Klawonn and Oliver Rheinbach. A hybrid approach to 3-level FETI. *PAMM Proc. Appl. Math. Mech.*, 8(1):10841–10843, 2008.
- [14] Axel Klawonn and Oliver Rheinbach. Highly scalable parallel domain decomposition methods with an application to biomechanics. *ZAMM Z. Angew. Math. Mech.*, 90(1):5–32, 2010.
- [15] Axel Klawonn and Olof B. Widlund. A domain decomposition method with Lagrange multipliers and inexact solvers for linear elasticity. *SIAM J. Sci. Comput.*, 22(4):1199–1219, 2000.
- [16] Axel Klawonn and Olof B. Widlund. Dual-primal FETI methods for linear elasticity. *Comm. Pure Appl. Math.*, 59(11):1523–1572, 2006.
- [17] Axel Klawonn, Olof B. Widlund, and Maksymilian Dryja. Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients. *SIAM J. Numer. Anal.*, 40(1):159–179, 2002.
- [18] Jungho Lee. Convergence analysis of a new domain decomposition method for a linearized contact problem. 2010. Submitted to *SIAM J. Sci. Comput.*
- [19] Jungho Lee. A strategy of finding an initial active set for inequality constrained quadratic programming problems. 2010. Submitted to *Optimization Methods and Software*.

- [20] Junggho Lee. *A Hybrid Domain Decomposition Method and its Applications to Contact Problems*. PhD thesis, Courant Institute of Mathematical Sciences, September 2009.
- [21] Jing Li and Olof B. Widlund. FETI-DP, BDDC, and block Cholesky methods. *Internat. J. Numer. Methods Engrg.*, 66(2):250–271, 2006.
- [22] Jan Mandel and Clark R. Dohrmann. Convergence of a balancing domain decomposition by constraints and energy minimization. *Numer. Linear Algebra Appl.*, 10(7):639–659, 2003. Dedicated to the 70th birthday of Ivo Marek.
- [23] Jan Mandel, Clark R. Dohrmann, and Radek Tezaur. An algebraic theory for primal and dual substructuring methods by constraints. *Appl. Numer. Math.*, 54(2):167–193, 2005.
- [24] Jan Mandel and Radek Tezaur. Convergence of a substructuring method with Lagrange multipliers. *Numer. Math.*, 73(4):473–487, 1996.
- [25] Jan Mandel and Radek Tezaur. On the convergence of a dual-primal substructuring method. *Numer. Math.*, 88(3):543–558, 2001.
- [26] Andrea Toselli and Olof Widlund. *Domain decomposition methods—algorithms and theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2005.
- [27] Xuemin Tu. Three-level BDDC in two dimensions. *Internat. J. Numer. Methods Engrg.*, 69(1):33–59, 2007.