Convergence analysis of a new domain decomposition method for a linearized contact problem

Jungho Lee*
Computer Science and Mathematics Division
Oak Ridge National Laboratory

January 21, 2010

Abstract

We analyze the condition number of a dual domain decomposition method which combines the original Finite Element Tearing and Interconnecting (FETI) and the dual-primal FETI (FETI-DP) method. This algorithm has been adopted by the engineering community for solving contact problems, which involve multiple bodies each of which has many degrees of freedom and therefore is decomposed into several subdomains. In this paper[1], we present a proof that such a method has a condition number which depends linearly on the number of subdomains across each body and logarithmically on the number of element across each subdomain. Our numerical results, and those of others, suggest that this is the best possible bound.

1 Introduction

The original FETI method, which later became to be known as the one-level FETI method, was first introduced by Farhat and Roux [15]. In this method, the domain is decomposed into nonoverlapping subdomains. The discretization of an elliptic partial differential equation, with continuity conditions across subdomain boundaries, is formulated as a Karush-Kuhn-Tucker (KKT) system with the displacement vectors as primal unknowns and the Lagrange multipliers as dual unknowns. The KKT system is reduced to an equation in terms of the Lagrange multipliers alone; this reduction process involves solving a Neumann problem on each subdomain. The resulting equation is solved using the conjugate gradient method. This original FETI method is scalable, in the sense that the rate of convergence is independent of the size of subdomains.

Later Farhat, Mandel and Roux introduced a variant of the FETI method [14] with a *Dirichlet preconditioner*, in which additionally Dirichlet problem is solved exactly on each subdomain in the preconditioning step. The use of this preconditioner makes the resulting algorithm even less sensitive to the number of degrees of freedoms in each subdomain. For further results and references, see, e.g., [16, 11, 2, 27, 22, 30].

A second generation of the FETI methods, namely the dual-primal FETI (FETI-DP) method, was introduced for linear elasticity problems in the plane by Farhat, Lesoinne, Le Tallec, Pierson and Rixen in [13]. In this method, a certain degree of the continuity coupling between subdomains, also known as *primal constraints*, is introduced; the continuity of the displacement vectors at some select interface nodes is built into the problem formulation, as in *primal* methods, whereas the continuity at other interface nodes is imposed by the use of *dual* Lagrange multipliers as in the one-level FETI method; thus the name *dual-primal* FETI. In FETI-DP methods, sufficiently many primal constraints are introduced so

---

[*] This author's work was supported in part by the U.S. Department of Energy under contracts DE-FG02-06ER25718 and DE-FC02-01ER25482 and in part by National Science Foundation grant DMS-0513251. The submitted manuscript has been authored by a contractor of the U.S. Government under Contract No. DE-AC05-00OR22725. Accordingly, the U.S. Government retains a non-exclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes.

[1] The results of this paper have already appeared in the Ph.D. thesis of the author, see [25].

that the local problems as well as the resulting stiffness matrix for the entire system become invertible. In addition, these primal constraints also provide a coarse solver which is needed for the scalability of the algorithm. Both one-level FETI and FETI-DP methods, equipped with a Dirichlet preconditioner, have condition number estimates of the form $C(1 + log(H/h))^2$, where H and h are the diameters of a typical subdomain and a typical element, respectively, and $C > 0$ is a constant independent of H and h. For a review of the history of various FETI-DP methods, see e.g., [17, 30, 23] and the references therein. The FETI-DP methods are preferred to the original one-level FETI methods for several reasons. Among them, a major advantage is that the introduction of the primal continuity constraints eliminates the need to solve singular problems; solving singular subproblems is not a trivial matter, see [12]. Also, the FETI-DP method allows us great flexibility in choosing the coarse solver, and this can make the algorithm robust with respect to the PDE coefficients. For further discussion and experimental results, see [17, 18].

In this paper, we combine the one-level FETI and the FETI-DP methods; we call the resulting method the FETI-FETI method. We consider bodies each of which has many degrees of freedom and is decomposed into several subdomains, which in turn are the unions of elements. Conversely, we can also think of a single body decomposed into many subdomains, which are grouped into *subclusters*; see [19], [20]. The continuity between subdomains of the same body (subcluster) is imposed in the manner of the FETI-DP method (Lagrange multipliers *and* primal constraints), whereas the continuity between different bodies is imposed purely by the use of Lagrange multipliers. Such an approach has been used in the context of contact problems; see [1] and [7, Chapter 8]. The same method was named a hybrid FETI-1/FETI-DP method in [19], [20].

In this paper, we prove that the condition number of the FETI-FETI method depends linearly on the number of subdomains across each body and logarithmically on the number of elements across each subdomain. Such results were previously shown numerically in [19] and [20], without a proof.

This paper is organized as follows; in Section 2, we review the one-level FETI and the FETI-DP methods. In Section 3, we introduce the FETI-FETI method and analyze its condition number estimate. Finally, in Section 4, we provide some numerical results which are consistent with the theory of Section 3.

2 Building blocks of the FETI-FETI method

2.1 A model problem and notation

We consider a second-order scalar elliptic problem on a bounded domain $\Omega \subset \mathbb{R}^n, n = 2, 3$. We denote the boundary of Ω by $\partial\Omega$, and assume that homogeneous Dirichlet boundary conditions are imposed on $\partial\Omega_D \subset \partial\Omega$, which is a subset of $\partial\Omega$ with a positive measure. Let $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_D$ be its complement. The corresponding Sobolev space in which the solution will be found is $H_0^1(\Omega, \partial\Omega_D) := \{v \in H^1(\Omega) : u = 0 \quad \text{on} \quad \partial\Omega_D\}$. We find $u \in H_0^1(\Omega, \partial\Omega_D)$ such that

$$a(u, v) = f(v), \quad \forall v \in H_0^1(\Omega, \partial\Omega_D), \tag{2.1}$$

where

$$a(u, v) := \int_\Omega \rho(x)\nabla u \cdot \nabla v, \quad f(v) = \int_\Omega fv. \tag{2.2}$$

Note that (2.1) is equivalent to the following minimization problem:

$$\min_{u \in H_0^1(\Omega, \partial\Omega_D)} \frac{1}{2}a(u, u) - f(u). \tag{2.3}$$

We decompose Ω into N nonoverlapping subdomains $\Omega_i, i = 1, \cdots, N$, each of which is the union of shape-regular elements. The diameter of Ω_i is H_i, and we interpret H_i/h_i as the ratio of H_i to the minimum mesh diameter in Ω_i. We note that many of the estimates in this paper will be expressed in terms of the ratio H/h, which is to be interpreted as $\max_i H_i/h_i$. We also note that $(H_i/h_i)^n, \Omega_i \subset \mathbb{R}^n$ gives a measure of the number of degrees of freedom associated with Ω_i.

The finite element nodes on the boundaries of neighboring subdomains match across the interface $\Gamma := \cup_{i \neq j} \partial\Omega_i \cap \partial\Omega_j$. Γ is the union of

- faces, edges and vertices in three dimensions: faces, regarded as open subsets of Γ, are shared by two subdomains. Edges, regarded as open subsets of the boundaries of the faces, are shared by more than two subdomains. Vertices are endpoints of edges.

- edges and vertices in two dimensions: edges, regarded as open subsets of Γ, are shared by two subdomains. Vertices, as in three dimensions, are endpoints of edges.

We note that the nodal values on $\partial\Omega_D$ will always vanish and those on $\partial\Omega_N$ which belong to only one subdomain will effectively belong to the subdomain interior. They will be eliminated together with the interior degrees of freedom when the given linear system is reduced to a Schur complement system associated with the interface Γ.

We assume that $\rho(x) = \rho_i \geq \rho_{min} > 0, \forall x \in \Omega_i, i = 1, \cdots, N$. We also introduce the corresponding set of interface nodes $\Gamma_h := \cup_{i \neq j} \partial\Omega_{i,h} \cap \partial\Omega_{j,h}$, where $\partial\Omega_{i,h}$ and $\partial\Omega_{j,h}$ are the sets of finite element nodes on $\partial\Omega_i$ and $\partial\Omega_j$, respectively. We also define local bilinear forms and linear functionals,

$$a^{(i)}(u,v) := \int_{\Omega_i} \rho(x) \nabla u \cdot \nabla v, \quad f^{(i)}(v) := \int_{\Omega_i} fv, \quad i = 1, \cdots, N. \tag{2.4}$$

In the rest of this section, we discuss the choice of the space of finite element functions in one-level FETI and FETI-DP methods.

We denote a standard finite element space of continuous, piecewise linear functions on Ω_i by $W^{(i)}$. We will always assume that these functions vanish on $\partial\Omega_D$. Each $W^{(i)}$ is decomposed into a subdomain interior part $W_I^{(i)}$ and a subdomain interface part $W_\Gamma^{(i)}$:

$$W^{(i)} = W_I^{(i)} \oplus W_\Gamma^{(i)}.$$

We denote the associated product spaces by $W := \prod_{i=1}^N W^{(i)}, W_I := \prod_{i=1}^N W_I^{(i)}$, and $W_\Gamma := \prod_{i=1}^N W_\Gamma^{(i)}$.

The functions in W and W_Γ are in general discontinuous across the interface, whereas the finite element solutions are continuous across the interface Γ. Therefore we introduce $\widehat{W}$ and $\widehat{W}_\Gamma$, which are the continuous subspaces of W and W_Γ, respectively.

For the FETI-DP methods, we will also need a subspace $\widetilde{W} \subset W$, intermediate between W and $\widehat{W}$, which consists of finite element functions which satisfy certain continuity constraints. The corresponding interface space is denoted by $\widetilde{W}_\Gamma$. In the two-dimensional case, we require the functions in $\widetilde{W}$ to be continuous at all subdomain vertices. In the three-dimensional case, enforcing such vertex constraints alone makes the condition number of the resulting algorithm quite sensitive to the number of degrees of freedom of each subdomain and we need different continuity constraints to obtain a better algorithm. However, the description of such continuity constraints is not the main focus of this paper and we refer the interested reader to [23], [17], and [18] for the three-dimensional case, for the sake of brevity.

We introduce the following decomposition of $\widetilde{W}_\Gamma$:

$$\widetilde{W}_\Gamma = W_\Delta \oplus \widehat{W}_\Pi = \left(\prod_{i=1}^N W_\Delta^{(i)} \right) \oplus \widehat{W}_\Pi,$$

where $\widehat{W}_\Pi$, a *primal subspace*, consists of continuous functions, and $W_\Delta^{(i)}$, a *dual subspace*, consists of functions which are allowed to be discontinuous across the interface. More precisely, $\widehat{W}_\Pi$ is spanned by subdomain vertex nodal basis functions, i.e., consists of functions which are nonzero on Γ only at subdomain vertices (*primal nodes*), in the two-dimensional case. Accordingly, $W_\Delta^{(i)} \in W_\Gamma^{(i)}$ consist of functions which are zero at the vertices of the subdomain Ω_i. The terminologies *primal* and *dual* indicate the fact that the continuity is imposed in the manner of *primal methods* at the *primal* nodes, and in the manner of *dual methods* (i.e., via Lagrange multipliers) at the *dual* nodes, respectively. In the three-dimensional case, we need to be more careful about the design of $\widehat{W}_\Pi$, i.e., the choice of *primal* constraints and $W_\Delta^{(i)}, i = 1, \cdots, N$, for the reasons mentioned above. W_Δ is the product space of $W_\Delta^{(i)}, i = 1, \cdots, N$ and we also define $W_\Pi = \prod_{i=1}^N W_\Pi^{(i)}$, where $W_\Pi^{(i)}$ is the local subspace of $\widehat{W}_\Pi$ for the subdomain $\Omega_i, i = 1, \cdots, N$. See Figure 1 for a depiction of $W, \widetilde{W}$, and $\widehat{W}$ in a two-dimensional case.

We note that we will not distinguish between a finite element function and its vector counterpart of nodal values.

For each subdomain $\Omega_i, i = 1, \cdots, N$, we assemble local stiffness matrices

$$A^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{\Gamma I}^{(i)^T} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix}$$

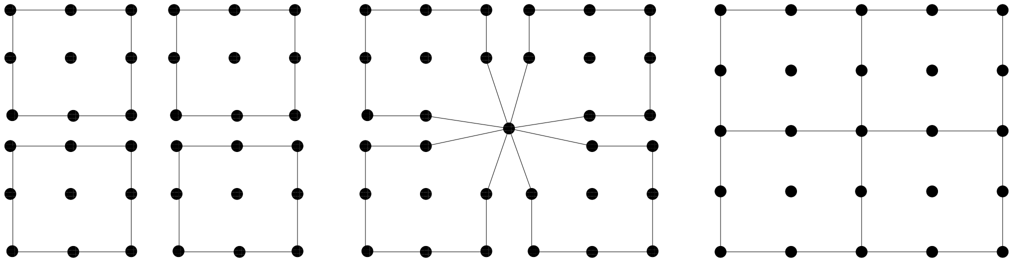
and local load vectors $f^{(i)}$ by integrating appropriate expressions over individual subdomains.

We also introduce scaling factors $\delta_i^\dagger(x)$ for each node $x \in \Gamma_h \cap \partial\Omega_{i,h}, i = 1, \cdots, N$: for $\gamma \in [1/2, \infty)$,

$$\delta_i^\dagger(x) = \frac{\rho_i^\gamma}{\sum_{j \in \mathcal{N}_x} \rho_j^\gamma}, \quad x \in \partial\Omega_{i,h} \cap \Gamma_h.$$

Here, $\mathcal{N}_x$ is the set of indices j of the subdomains such that $x \in \partial\Omega_{j,h}$.

Figure 1: subdomain couplings for $W, \widetilde{W}$, and $\widetilde{W}$



(a) W: One-Level (b) $\widetilde{W}$: FETI-DP (c) $\widehat{W}$

2.2 The One-Level FETI method

In this subsection, we review the one-level FETI method, following [30, Section 6.3]. We use the finite element functions in the space W to discretize the minimization problem (2.3). Since the functions in W are in general discontinuous across the interface Γ, we need to enforce the continuity condition explicitly:

$$\min_{u \in W} \frac{1}{2} a(u,u) - f(u), \quad \text{subject to} \quad Bu = 0. \tag{2.5}$$

$Bu = 0$ represents continuity constraints across the interface Γ, where

$$B = [B^{(1)}, B^{(2)}, \cdots, B^{(N)}]$$

is a matrix consisting of elements $0, 1, -1$ such that $Bu = 0$ if and only if all the values of u associated with more than one subdomain boundary coincide. The columns of $B^{(i)}$ which correspond to the interior nodes of Ω_i are zero. Thus, $B^{(i)} = [\ 0 \quad B_\Gamma^{(i)}\ ]$ when the interior degrees of freedom are ordered first. We call B a jump operator. We can rewrite the minimization problem (2.5) using the matrix notation:

$$\min_{u \in W} \frac{1}{2} u^T A u - f^T u, \quad \text{subject to} \quad Bu = 0, \tag{2.6}$$

where

$$A = \begin{bmatrix} A^{(1)} & & \\ & \ddots & \\ & & A^{(N)} \end{bmatrix}, \quad f = \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(N)} \end{bmatrix}.$$

Introducing a vector of Lagrange multipliers λ to enforce the continuity constraint $Bu = 0$, we obtain the following Karush-Kuhn-Tucker (KKT) system:

Find $(u, \lambda) \in W \times range(B)$, such that

$$\begin{array}{lclcl} Au & + & B^T\lambda & = & f \\ Bu & & & = & 0 \end{array}. \tag{2.7}$$

λ is unique only up to an additive element of $\ker(B^T)$. The space of Lagrange multipliers, U, is therefore chosen as range(B).

Eliminating the interior unknowns in each subdomain, we obtain the following:

Find $(u_\Gamma, \lambda) \in W_\Gamma \times range(B_\Gamma)$, such that

$$\begin{array}{lclcl} Su_\Gamma & + & B_\Gamma^T\lambda & = & g \\ B_\Gamma u_\Gamma & & & = & 0 \end{array}, \tag{2.8}$$

where

$$S = \begin{bmatrix} S^{(1)} & & \\ & \ddots & \\ & & S^{(N)} \end{bmatrix}, \quad S^{(i)} = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} A_{\Gamma I}^{(i)^T}, i = 1, \cdots, N,$$

$$g = \begin{bmatrix} g^{(1)} \\ \vdots \\ g^{(N)} \end{bmatrix}, \quad g^{(i)} = f_\Gamma^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} f_I^{(i)}, i = 1, \cdots, N$$

and $B_\Gamma = [B_\Gamma^{(1)}, B_\Gamma^{(2)}, \cdots, B_\Gamma^{(N)}]$ is obtained by removing the zero columns of B that correspond to the interior nodes of individual subdomains, resulting in $Bu = B_\Gamma u_\Gamma$ where $B = [\ 0 \quad B_\Gamma\ ]$ and $u^T = [\ u_I^T \quad u_\Gamma^T\ ]$.

In all FETI methods, we reduce the KKT system (2.8) to an equation of λ alone, by solving the first equation of (2.8) for u_Γ. The matrices A in (2.7) and S in (2.8), however, are generally singular, when there are subdomains with boundaries which do not intersect the Dirichlet boundary $\partial\Omega_D$. We call such subdomains floating. In such a case the solution of the first equation of (2.8) exists if and only if $g - B_\Gamma^T\lambda \in \text{range}(S)$; this requirement leads to the introduction of a projection P, which will be introduced shortly. First, we introduce a matrix R such that range$(R) = \ker(S)$:

$$R = \begin{bmatrix} R^{(1)} & & \\ & \ddots & \\ & & R^{(N)} \end{bmatrix},$$

where $R^{(i)}$ consists of the null vectors of $S^{(i)}, i = 1, \cdots, N$. Subdomains with nonsingular stiffness matrices do not contribute to the matrix R, i.e., $R^{(i)}$ is an empty matrix if the subdomain Ω_i intersects the Dirichlet boundary $\partial\Omega_D$. We now solve the first equation of (2.8) for u_Γ:

$$u_\Gamma = S^\dagger(g - B_\Gamma^T\lambda) + R\alpha \quad \text{if} \quad g - B_\Gamma^T\lambda \in \text{range}(S) = \ker(S)^\perp = \text{range}(R)^\perp, \tag{2.9}$$

where $S^\dagger$ is a pseudoinverse of S and α has to be determined. Substituting (2.9) into the second equation of (2.8), we obtain

$$B_\Gamma S^\dagger B_\Gamma^T\lambda = B_\Gamma S^\dagger g + B_\Gamma R\alpha. \tag{2.10}$$

We introduce the notation $F := B_\Gamma S^\dagger B_\Gamma^T, d := B_\Gamma S^\dagger g, G := B_\Gamma R, e := R^T g$ and $P := I - G(G^TG)^{-1}G^T$. Note that P is a projection operator with its range orthogonal to G. We apply this P to (2.10) to eliminate the term with α and rewrite the orthogonality condition in (2.9) to obtain the following:

$$\left\{ \begin{array}{rcl} PF\lambda & = & Pd \\ G^T\lambda & = & e. \end{array} \right. \tag{2.11}$$

We define the space

$$V := \{\mu \in U : B_\Gamma^T \mu \in \mathrm{range}(S)\} = \ker(G^T),$$

which we call the space of admissible increments, following Chen and Mandel [10]. The one-level FETI method is a preconditioned conjugate gradient method applied to

$$PF\lambda = Pd, \quad \lambda \in \lambda_0 + V \tag{2.12}$$

where λ_0 is chosen such that $G^T\lambda_0 = e$. Here, we only consider the *Dirichlet preconditioner* $M_D^{-1} := B_{D,\Gamma} S B_{D,\Gamma}^T$, where $B_{D,\Gamma} = [B_{D,\Gamma}^{(1)} \cdots B_{D,\Gamma}^{N}]$ is a scaled jump operator. $B_{D,\Gamma}^{(i)}$ is obtained as follows: each nonzero entry of $B_\Gamma^{(i)}$ contributes to the Lagrange multiplier enforcing the continuity at a node $x \in \partial\Omega_i \cap \partial\Omega_j$ and is multiplied by $\delta_j^\dagger(x)$ to produce the corresponding $B_{D,\Gamma}^{(i)}$.

With this choice of preconditioner, the preconditioned operator of the one-level FETI method has the following condition number bound:

$$\mathcal{K} \le C(1 + log(H/h))^2, \tag{2.13}$$

where $\mathcal{K}$ denotes the condition number of the preconditioned operator in the appropriate subspace. For a proof of (2.13), see [27] or [30, Section 6.3]. Thus the convergence rate of the one-level FETI method depends only polylogarithmically on the number of degrees of freedom of a subdomain.

2.3 The FETI-DP method

In this subsection, we closely follow the notation of [26]. For more details on various FETI-DP methods, see, e.g., [26, 23, 17, 18, 30] and the references therein. In the FETI-DP method, we use finite element functions in $\widetilde{W} = W_I \oplus \widetilde{W}_\Gamma$ to discretize (2.3).

We first note that the local stiffness matrices $A^{(i)}$ and the local load vectors $f^{(i)}$ can be written as follows:

$$A^{(i)} = \begin{bmatrix} A_{II}^{(i)} & {A_{\Delta I}^{(i)}}^T & {A_{\Pi I}^{(i)}}^T \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & {A_{\Pi I}^{(i)}}^T \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{bmatrix}, \quad f^{(i)} = \begin{bmatrix} f_I^{(i)} \\ f_\Delta^{(i)} \\ f_\Pi^{(i)} \end{bmatrix}, \tag{2.14}$$

where I, Δ, and Π indicate the index sets corresponding to the interior nodes, dual nodes, i.e., those of $W_\Delta^{(i)}$, and primal nodes, i.e., those of $W_\Pi^{(i)}$, respectively. We introduce the matrix $\widetilde{A}$, which can be thought of as the restriction of A, defined for the functions in W, to the subspace $\widetilde{W}$:

$$\widetilde{A} = \begin{bmatrix} A_{II}^{(1)} & {A_{\Delta I}^{(1)}}^T & & & & {\widetilde{A}_{\Pi I}^{(1)}}^T \\ A_{\Delta I}^{(1)} & A_{\Delta\Delta}^{(1)} & & & & {\widetilde{A}_{\Pi\Delta}^{(1)}}^T \\ & & \ddots & & & \vdots \\ & & & A_{II}^{(N)} & {A_{\Delta I}^{(N)}}^T & {\widetilde{A}_{\Pi I}^{(N)}}^T \\ & & & A_{\Delta I}^{(N)} & A_{\Delta\Delta}^{(N)} & {\widetilde{A}_{\Pi\Delta}^{(N)}}^T \\ \widetilde{A}_{\Pi I}^{(1)} & \widetilde{A}_{\Pi\Delta}^{(1)} & \cdots & \widetilde{A}_{\Pi I}^{(N)} & \widetilde{A}_{\Pi\Delta}^{(N)} & \widetilde{A}_{\Pi\Pi} \end{bmatrix}. \tag{2.15}$$

Here,

$$\widetilde{A}_{\Pi I}^{(i)} = {R_\Pi^{(i)}}^T A_{\Pi I}^{(i)}, \quad \widetilde{A}_{\Pi\Delta}^{(i)} = {R_\Pi^{(i)}}^T A_{\Pi\Delta}^{(i)}, \quad i = 1, \cdots, N,$$

and

$$\widetilde{A}_{\Pi\Pi} = \sum_{i=1}^{N} {R_\Pi^{(i)}}^T A_{\Pi\Pi}^{(i)} R_\Pi^{(i)},$$

where $R_{\Pi}^{(i)} : \widehat{W}_{\Pi} \longrightarrow W_{\Pi}^{(i)}, i = 1, \cdots, N$, is a restriction operator which extracts the relevant subdomain component belonging to $W_{\Pi}^{(i)}$ from a vector in $\widehat{W}_{\Pi}$. As in the one-level FETI method, we introduce a vector of Lagrange multipliers and obtain the following saddle point problem:

Find $(u, \lambda) \in \widetilde{W} \times range(\widetilde{B})$, such that

$$\begin{array}{lclcl} \widetilde{A}u & + & \widetilde{B}^T\lambda & = & f \\ \widetilde{B}u & & & = & 0 \end{array}. \tag{2.16}$$

Again, $\widetilde{B}$ is a jump operator such that $\widetilde{B}u = 0, u \in \widetilde{W}$ if and only if the values of u associated with more than one subdomain coincide. Eliminating the interior unknowns of each subdomain from the system (2.16), we obtain:

Find $(u, \lambda) \in \widetilde{W_\Gamma} \times range(\widetilde{B}_\Gamma)$, such that

$$\begin{array}{lclcl} \widetilde{S}_\Gamma u_\Gamma & + & \widetilde{B}_\Gamma^T\lambda & = & g \\ \widetilde{B}_\Gamma u & & & = & 0 \end{array}. \tag{2.17}$$

$\widetilde{S}_\Gamma$ can also be regarded as the restriction of S, defined on W_Γ, to the subspace $\widetilde{W_\Gamma}$:

$$\widetilde{S}_\Gamma = \widetilde{R}_\Gamma^T S \widetilde{R}_\Gamma,$$

where $\widetilde{R}_\Gamma : \widetilde{W_\Gamma} \longrightarrow W_\Gamma$ is a direct sum of restriction operators that extract the subdomain part belonging to $W_\Gamma^{(i)}$ from a vector in $\widetilde{W_\Gamma}$.

The matrix $\widetilde{A}$, and therefore also $\widetilde{S}_\Gamma$, are nonsingular, so we can solve the first equation of (2.17) for u_Γ without any difficulty and substitute the resulting equation into the second equation of (2.17):

$$\widetilde{B}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{B}_\Gamma^T \lambda = -\widetilde{B}_\Gamma \widetilde{S}_\Gamma^{-1} g. \tag{2.18}$$

The Dirichlet preconditioner used in the FETI-DP algorithms to solve the equation (2.18) is $\widetilde{B}_{D,\Gamma} \widetilde{S}_\Gamma \widetilde{B}_{D,\Gamma}^T$. $\widetilde{B}_{D,\Gamma} = [\widetilde{B}_{D,\Gamma}^{(1)}, \cdots, \widetilde{B}_{D,\Gamma}^{(N)}]$ is obtained in exactly the same manner as $B_{D,\Gamma}$ in Section 2.2.

With this choice of preconditioner, the preconditioned operator for the FETI-DP method also has the condition number bound (2.13). For a proof of this convergence bound for the two-dimensional case, see, e.g., [28]. For three-dimensional scalar elliptic problems and linear elasticity problems, see [24] and [23], respectively.

3 A FETI-FETI Method

3.1 Motivation and Notations

Our ultimate goal is to solve contact problems with N bodies, $\Omega_1, \cdots, \Omega_N$. Contact problems are characterized by an active area of contact, which is unknown a priori, and inequality constraints such as non-penentration conditions; see [1], [31]. We recall that the subdomain interface continuity constraints are of the form $Bu = 0$ in the FETI methods, which are due to the use of a domain decomposition algorithm and the fact that finite element functions that are used are not continuous across the interface. The introduction of the subdomains and the ensuing need for continuity constraints such as $Bu = 0$ are artificial in a sense. In contact problems, however, inequality constraints arise from the fact that we have multiple bodies and they are inherent to the problem.

In the rest of this paper, we only consider the case of matching grids between bodies. We also assume that we use an active set method to deal with the inequality constraints; for other ways of dealing with inequality constraints, see [1], [31]. An active set method gives rise to a sequence of auxiliary equality constrained problems, in which some of the inequality constraints are replaced by corresponding equality constraints and the rest are ignored. An active set method has outer iterations in which the active set is updated, and a minimization problem on the current active face is solved in each inner iteration. The FETI-FETI method concerns the inner minimization problem.

In this paper, we concentrate on scalar elliptic problems in two- and three-dimensions with inequality constraints. We present the following model problem as a motivation; it is taken from [7, Chapter 8]:

$$\min \sum_{i=1}^{2} \left(\frac{1}{2} \int_{\Omega^i} |\nabla u^i|^2 dx - \int_{\Omega^i} f u^i dx \right) \tag{3.1}$$

where

$$\begin{array}{lcl} u^i & \in & H^1(\Omega^i), i = 1, 2, \quad \Omega^1 = (0,1) \times (0,1), \Omega^2 = (1,2) \times (0,1) \\ u^1 & = & 0 \quad \text{on} \quad \Gamma_u^1 = \{0\} \times (0,1) \\ u^2 & - & u^1 \geq 0 \quad \text{on} \quad \Gamma_c = \{1\} \times (0,1) \end{array} \tag{3.2}$$

and

$$f \equiv \begin{cases} -3 & \text{on} \quad (0,1) \times (0.75,1) \\ -1 & \text{on} \quad (1,2) \times (0,0.25) \\ 0 & \text{on} \quad (0,1) \times (0,0.75) \cup (1,2) \times (0.25,1) \end{cases} \tag{3.3}$$

The reason that we consider only scalar elliptic problems is that the inequality constraints in scalar elliptic problems are much simpler than those in linear elasticity problems and their simplicity allows us to focus on the analysis of the preconditioned operator. In linear elasticity problems, the non-penentration conditions depend on the current configuration of the bodies and need to be updated in each iteration (see [1, Section 4], [31]), whereas in this scalar problem the inequality condition is expresssed by a single equation such as $Bu \leq 0$ throughout.

We consider multiple bodies $\Omega_i, i = 1, \cdots, N$, each of which has many degrees of freedom and is decomposed into subdomains $\Omega_{i,j}, i = 1, \cdots, N, j = 1, \cdots, N_i$. The diameter of the body Ω_i is H_i^b, with $H_b = \max_i H_i^b$. The diameter of the subdomain $\Omega_{i,j}$ is $H_{i,j}^s$, with $H_s = \max_{i,j} H_{i,j}^s$. However, H_i^b/H_i^s is to be interpreted as the ratio of H_i^b to the diameter of the smallest subdomain of Ω_i, and we let $H_b/H_s := max_i H_i^b/H_i^s$. When there is no danger of confusion, we will use the notation H instead of H_s as in Section 2.1.

We assume that at least one body is clamped on part of its boundary, and denote the union of such fixed boundaries for the entire system by $\partial\Omega_D$. We assume that $\rho(x) = \rho_{i,j} \geq \rho_{min} > 0, \forall x \in \Omega_{i,j}, \forall i, j$, and also,

$$\rho_i := \max_j \rho_{i,j} \leq C \rho_{i,j}, \quad \forall i, j, \tag{3.4}$$

where $C \geq 1$ is a constant independent of i.

We introduce two types of global interfaces: the first one is $\overline{\Gamma_{gl}} := \bigcup_{i \neq j} \partial\Omega_i \cap \partial\Omega_j$, and can be viewed as the potential contact area between the bodies: in the model problem, this is Γ_c. The second one, the *current* contact area, is denoted by Γ_{gl}^k, where $\Gamma_{gl}^k \subset \overline{\Gamma_{gl}}$; the superscript k concerns the outer iteration of the active set method and reminds us that the current active set/area changes. In each outer iteration of the active set method some of the inequality constraints are adopted as the corresponding equality constraints and the rest are ignored, and $\Gamma_{gl,h}^k$, the discrete version of Γ_{gl}^k, can be viewed as the collection of the nodes at which equality constraints are being imposed. We also introduce the local interfaces $\Gamma_{loc}^{(i)} := \bigcup_{j \neq k} (\partial\Omega_{i,j} \cap \partial\Omega_{i,k}), i = 1, \cdots, N$. We assume there are no traction forces and denote the union of the free boundaries by $\partial\Omega_F := (\cup_i \partial\Omega_i) \setminus (\partial\Omega_D \cup \overline{\Gamma_{gl}})$.

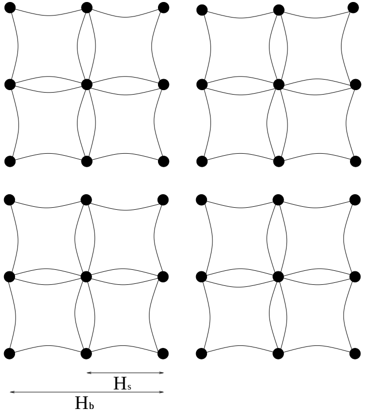
We denote the standard finite element space of continuous, piecewise linear functions on $\Omega_{i,j}$ by $W^{(i,j)}$. Each $W^{(i,j)}$ is decomposed into a subdomain interior part $W_I^{(i,j)}$ and a subdomain interface part $W_\Gamma^{(i,j)}$ for functions on $\partial\Omega_{i,j} \cap \overline{\Gamma_{gl}}$. $W_\Gamma^{(i,j)}$ is further decomposed into a primal subspace $W_\Pi^{(i,j)}$ and a dual subspace $W_\Delta^{(i,j)}$ in the style of the FETI-DP method. In the two-dimensional case, which is the focus of our analysis in this paper, we can impose the primal continuity at all vertex nodes. We also note that all functions vanish on the Dirichlet boundary $\partial\Omega_D$. We define associated product spaces, $W_I^{(i)} := \prod_{j=1}^{N_i} W_I^{(i,j)}, W_\Gamma^{(i)} := \prod_{j=1}^{N_i} W_\Gamma^{(i,j)}, W_\Delta^{(i)} := \prod_{j=1}^{N_i} W_\Delta^{(i,j)}, W_\Pi^{(i)} := \prod_{j=1}^{N_i} W_\Pi^{(i,j)}$, and $\widehat{W}_\Pi^{(i)}$, which is the continuous subspace of $W_\Pi^{(i)}$.

We introduce spaces analogous to $\widetilde{W}_\Gamma$ and $\widehat{W}_\Gamma$ of Section 2. Functions in $W_\Gamma^{(i)}$ are in general discontinuous across the local interface $\Gamma_{loc}^{(i)}$, and we define $\widehat{W}_\Gamma^{(i)}$ as the continuous subspace of $W_\Gamma^{(i)}$.

$\widetilde{W}_{\Gamma}^{(i)} := W_{\Delta}^{(i)} \oplus \widehat{W}_{\Pi}^{(i)}$ is intermediate between $\widehat{W}_{\Gamma}^{(i)}$ and $W_{\Gamma}^{(i)}$. Also, we let $\widetilde{W}^{(i)} := W_I^{(i)} \oplus \widetilde{W}_{\Gamma}^{(i)}$.

In the formulation of the FETI-FETI method in two dimensions, we will use the spaces $\widetilde{W}_c := \prod_{i=1}^N \widetilde{W}^{(i)}$ and $\widetilde{W}_{\Gamma,c} := \prod_{i=1}^N \widetilde{W}_{\Gamma}^{(i)}$. See Figure 2.

Figure 2: $\widetilde{W}_c$ for FETI-FETI



We introduce the matrices $A^{(i)}$, which are direct sums of the stiffness matrices $A^{(i,j)}, j = 1, \cdots, N_i$, for individual subdomains:

$$A^{(i)} = \begin{bmatrix} A^{(i,1)} & & \\ & \ddots & \\ & & A^{(i,N_i)} \end{bmatrix}, \quad i = 1, \cdots, N. \tag{3.5}$$

We also introduce the matrices $\widetilde{A}^{(i)}$, which are analogous to the matrix $\widetilde{A}$ introduced in (2.15). They are the restrictions of $A^{(i)}$ to $\widetilde{W}^{(i)}$:

$$\widetilde{A}^{(i)} = \begin{bmatrix} A_{II}^{(i,1)} & A_{\Delta I}^{(i,1)^T} & & & & \widetilde{A}_{\Pi I}^{(i,1)^T} \\ A_{\Delta I}^{(i,1)} & A_{\Delta\Delta}^{(i,1)} & & & & \widetilde{A}_{\Pi\Delta}^{(i,1)^T} \\ & & \ddots & & & \vdots \\ & & & A_{II}^{(i,N_i)} & A_{\Delta I}^{(i,N_i)^T} & \widetilde{A}_{\Pi I}^{(i,N_i)^T} \\ & & & A_{\Delta I}^{(i,N_i)} & A_{\Delta\Delta}^{(i,N_i)} & \widetilde{A}_{\Pi\Delta}^{(i,N_i)^T} \\ \widetilde{A}_{\Pi I}^{(i,1)} & \widetilde{A}_{\Pi\Delta}^{(i,1)} & \cdots & \widetilde{A}_{\Pi I}^{(i,N_i)} & \widetilde{A}_{\Pi\Delta}^{(i,N_i)} & \widetilde{A}_{\Pi\Pi}^{(i)} \end{bmatrix}.$$

Here,

$$\widetilde{A}_{\Pi I}^{(i,j)} = R_{\Pi}^{(i,j)^T} A_{\Pi I}^{(i,j)}, \quad \widetilde{A}_{\Pi\Delta}^{(i,j)} = R_{\Pi}^{(i,j)^T} A_{\Pi\Delta}^{(i,j)}, \quad i = 1, \cdots, N_i,$$

and

$$\widetilde{A}_{\Pi\Pi}^{(i)} = \sum_{j=1}^{N_i} R_{\Pi}^{(i,j)^T} A_{\Pi\Pi}^{(i,j)} R_{\Pi}^{(i,j)},$$

where $R_{\Pi}^{(i,j)} : \widehat{W}_{\Pi}^{(i)} \longrightarrow W_{\Pi}^{(i,j)}, j = 1, \cdots, N_i$ is a restriction operator similar to $R_{\Pi}^{(i)}$ of Section 2.3.
A Schur complement $\widetilde{S}_{\Gamma}^{(i)}$ on $\widetilde{W}_{\Gamma}^{(i)}$ is obtained by eliminating the interior unknowns of each subdomain from $\widetilde{A}^{(i)}$. Note that $\widetilde{S}_{\Gamma}^{(i)}$ can also be regarded as the restriction of $S^{(i)}$ to $\widetilde{W}_{\Gamma}^{(i)}$, i.e.,

$$\widetilde{S}_{\Gamma}^{(i)} = \widetilde{R}_{\Gamma}^{(i)^T} S^{(i)} \widetilde{R}_{\Gamma}^{(i)},$$

where

$$S^{(i)} = \begin{bmatrix} S^{(i,1)} & & \\ & \ddots & \\ & & S^{(i,N_i)} \end{bmatrix}, \quad S^{(i,j)} = A_{\Gamma\Gamma}^{(i,j)} - A_{\Gamma I}^{(i,j)^T} A_{II}^{(i,j)^{-1}} A_{\Gamma I}^{(i,j)^T}, \quad j = 1, \cdots, N_i,$$

and $\widetilde{R}_{\Gamma}^{(i)} : \widetilde{W}_{\Gamma}^{(i)} \to W_{\Gamma}^{(i)}$, similar to $\widetilde{R}_{\Gamma}$ of Section 2.3.

Recalling that we are using an active set method to deal with the inequality conditions, we formulate the minimization problem on the current active set:

$$\min_{u \in \widetilde{W}_c} \frac{1}{2} u^T \widetilde{A}_c u - \widetilde{f}_c^T u, \quad \text{with} \quad Z^k \widetilde{B}_c u = 0, \tag{3.6}$$

where

$$\widetilde{A}_c = \begin{bmatrix} \widetilde{A}^{(1)} & & \\ & \ddots & \\ & & \widetilde{A}^{(N)} \end{bmatrix}, \quad u = \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(N)} \end{bmatrix}, \quad \widetilde{f}_c = \begin{bmatrix} \widetilde{f}^{(1)} \\ \vdots \\ \widetilde{f}^{(N)} \end{bmatrix},$$

$$u^{(i)} \in \widetilde{W}^{(i)}, \quad i = 1, \cdots, N,$$

and

$$\widetilde{B}_c = \begin{bmatrix} B_{loc} \\ B_{gl} \end{bmatrix} = \begin{bmatrix} B_{loc}^{(1)} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & B_{loc}^{(N)} \\ B_{gl}^{(1)} & \cdots & B_{gl}^{(N)} \end{bmatrix},$$

$$B_{loc}^{(i)} = \begin{bmatrix} B_{loc}^{(i,1)} & \cdots B_{loc}^{(i,N_i)} \end{bmatrix}, \quad i = 1, \cdots, N,$$

$$Z^k = \begin{bmatrix} I & 0 \\ 0 & Z_{gl}^k \end{bmatrix}.$$

$Z^k \widetilde{B}_c u = 0$ in (3.6) indicates the continuity constraint across the local subdomain interface $\Gamma_{loc}^{(i)}, i = 1, \cdots, N$, as well as the continuity constraint across the global area of contact Γ_{gl}^k. Z_{gl}^k is a square matrix obtained by replacing some of the diagonal entries of the identity matrix with zeros; only the entries corresponding to the nodes at which an equality is imposed are retained. We use the superscript k to remind us that Z_{gl}^k and Z^k change in each iteration of the active set method. We have $B_{loc}^{(i)} u^{(i)} = 0, u^{(i)} \in \widetilde{W}^{(i)}$, exactly when the values associated with more than one subdomain on the body Ω_i coincide. Note that $B_{loc}^{(i)}$ has nonzero columns only for the components of $W_{\Delta}^{(i)}$.

We also introduce a scaled jump operator, $\widetilde{B}_{D,c}$:

$$\widetilde{B}_{D,c} = \begin{bmatrix} B_{loc,D} \\ B_{gl,D} \end{bmatrix} = \begin{bmatrix} B_{loc,D}^{(1)} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & B_{loc,D}^{(N)} \\ B_{gl,D}^{(1)} & \cdots & B_{gl,D}^{(N)} \end{bmatrix}$$

and

$$B_{loc,D}^{(i)} = \begin{bmatrix} B_{loc,D}^{(i,1)} & \cdots B_{loc,D}^{(i,N_i)} \end{bmatrix}, \quad i = 1, \cdots, N.$$

$B_{loc,D}^{(i)}$ and $B_{gl,D}^{(i)}$ are obtained in the same manner as $B_{D,\Gamma}$ of the one-level FETI method; see Section 2.2. The nonzero entry of $B_{loc}^{(i,j)}$ associated with the Lagrange multipliers for the continuity at the node $x \in \partial\Omega_{i,j} \cap \Omega_{i,k}$ is multiplied by $\delta_{i,k}^{\dagger}(x) = \rho_{i,k}^{\gamma}(x) / \sum_{s \in \mathcal{N}_{x,loc}^{(i)}} \rho_{i,s}^{\gamma}(x)$, where $\mathcal{N}_{x,loc}^{(i)}$ is the set of indices of the subdomains of Ω_i with x on their boundary. The nonzero entry of $B_{gl}^{(i)}$ associated with the Lagrange multiplier for the continuity at the node $x \in \partial\Omega_i \cap \partial\Omega_j$ is multiplied by $\delta_j^\dagger(x) = \sum_{s \in \mathcal{N}_{x,loc}^{(j)}} \rho_{j,s}^\gamma(x) / \sum_{k \in \mathcal{N}_{x,gl}, t \in \mathcal{N}_{x,loc}^{(k)}} \rho_{k,t}^\gamma(x)$, where $\mathcal{N}_{x,gl}$ is the set of indices of the subdomains of any body which share the node x on their boundary.

Eliminating the interior unknowns in all subdomains of each body, we obtain the following reduced minimization problem,

$$\min_{u_\Gamma \in \widetilde{W}_{\Gamma,c}} \frac{1}{2} u_\Gamma^T \widetilde{S}_c u_\Gamma - \widetilde{g}_c^T u_\Gamma, \quad \text{with} \quad Z_\Gamma^k \widetilde{B}_{\Gamma,c} u_\Gamma = 0, \tag{3.7}$$

where

$$\widetilde{S}_c = \begin{bmatrix} \widetilde{S}_\Gamma^{(1)} & & \\ & \ddots & \\ & & \widetilde{S}_\Gamma^{(N)} \end{bmatrix}, \quad u_\Gamma = \begin{bmatrix} u_\Gamma^{(1)} \\ \vdots \\ u_\Gamma^{(N)} \end{bmatrix}, \quad u_\Gamma^{(i)} \in \widetilde{W}_\Gamma^{(i)}, i = 1, \cdots, N.$$

Z_Γ^k is obtained by removing the rows and the columns of Z^k that correspond to the subdomain interior nodes. This minimization problem is equivalent to the following KKT system:

$$\begin{bmatrix} \widetilde{S}_c & (Z_\Gamma^k B_{\Gamma,c})^T \\ Z_\Gamma^k \widetilde{B}_{\Gamma,c} & 0 \end{bmatrix} \begin{bmatrix} u_\Gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} \widetilde{g}_c \\ 0 \end{bmatrix}. \tag{3.8}$$

It is natural to reduce this system to an equation for λ as in the one-level FETI method and solve it with the PCG method in a proper subspace, using the following preconditioner:

$$M_D^{-1} := Z_\Gamma^k \widetilde{B}_{D,\Gamma_c} \widetilde{S} \widetilde{B}_{D,\Gamma_c}^T Z_\Gamma^k.$$

The resulting method, which we name the FETI-FETI method, turns out not to be scalable with respect to the number of subdomains. We will present a partial explanation for this phenomenon in Section 3.4, using the framework of [30, Section 6.3].

Let $P_D := B_{D,\Gamma}^T B_\Gamma$, where B_Γ and $B_{D,\Gamma}$ are the jump operator and the scaled jump operator for the one-level FETI method, respectively, defined in Section 2.2. At the core of the eigenvalue analysis for the one-level FETI method is the following result:

Lemma 3.1. *For any $w \in \text{range}(S)$, we have*

$$|P_D w|_S^2 \le C(1 + log(H/h))^2 |w|_S^2,$$

where C is independent of H, h.

For a proof, see [30, Section 6.2.3]. This lemma is used for bounding $\lambda_{max}(M_D^{-1}F)$ from above. It is easy to show that $\lambda_{min}(M_D^{-1}F) \ge 1$, and therefore that the condition number of the preconditioned operator for the one-level FETI method grows like $C(1 + log(H/h))^2$. We analyze the condition number estimate of the FETI-FETI method with a similar technique, by bounding $|\widetilde{P}_D^k w|_{\widetilde{S}_c}$ from above by $|w|_{\widetilde{S}_c}$, for $w \in range(\widetilde{S}_c)$. Here, $\widetilde{P}_D^k := \widetilde{B}_{D,\Gamma_c}^T Z_\Gamma^k \widetilde{B}_{\Gamma_c}$. We will show that we do not obtain as benign a bound as in Lemma 3.1 for the FETI-FETI method. This result is supported by the numerical experiments of Section 4. We first need some technical tools.

3.2 Technical Tools, part I

In this subsection, we collect technical tools that are mostly from [30, Chapter 4], [6], and [21]. We note that our own lemmas in Section 3.3 and the analysis in Section 3.4 are currently only for Lipschitz subdomains, although some of the lemmas we introduce in this section, namely those taken from [6] and [21], are for more general subdomains. This is to inform the reader of the recent extension of the well known results in the theory of domain decomposition methods to less regular subdomains.

We first give the definition of John domains and uniform domains, the latter of which are also known as Jones domains. For a more detailed discussion, see [6], [21] and the references therein.

Definition 1 (John Domains). *A domain $\Omega \subset \mathbb{R}^n$ - an open, bounded, and connected set - is a John domain if there exist a constant C_J and a distinguished central point $x_0 \in \Omega$ such that each $x \in \Omega$ can be joined to it by a rectifiable curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x_0, \gamma(1) = x$ and $|x - \gamma(t)| \leq C_J \cdot distance(\gamma(t), \partial\Omega)$ for all $t \in [0,1]$.*

Definition 2 (Uniform Domains). *A domain $\Omega \subset \mathbb{R}^n$ is a uniform domain if there exists a constant C_U such that any pair of points $x_1 \in \Omega$ and $x_2 \in \Omega$ can be joined by a rectifiable curve $\gamma(t) : [0,1] \to \Omega$ with $\gamma(0) = x_1, \gamma(1) = x_2$ and the Euclidean arc length of $\gamma \leq C_U|x_1 - x_2|$ and $min_{i=1,2}|x_i - \gamma(t)| \leq C_U \cdot distance(\gamma(t), \partial\Omega)$ for all $t \in [0,1]$.*

The following is [21, Lemma 4.5].

Lemma 3.2 (Extension Lemma). *Let Ω_i and Ω_j be subsets of $\mathbb{R}^n$ and two subdomains with a common $(n-1)-$dimensional interface $\mathcal{E}^{ij}$. Furthermore, let Ω_i be a uniform domain, let $V_i^h = \{v_h \in W^h(\Omega_i) : v_h(x) = 0 \quad \text{at all nodes of } \partial\Omega_i \setminus \mathcal{E}^{ij}\}$, and let $V_j^h = \{v_h \in W^h(\Omega_j) : v_h(x) = 0 \quad \text{at all nodes of } \partial\Omega_j \setminus \mathcal{E}^{ij}\}$, where $W^h(\Omega_i)$ is the standard finite element space of continuous, piecewise linear functions on Ω_i. Then, there exists an extension operator*

$$E_{ji}^h : V_j^h \to V_i^h, \tag{3.9}$$

with the following properties:

$$\begin{array}{ll} 1. & (E_{ji}^h u_h)|_{\Omega_j} = u_h, \quad \forall u_h \in V_j^h \\ 2. & ||E_{ji}^h u_h||_{H^1(\Omega_i)} \leq C||u_h||_{H^1(\Omega_j)}, \quad \forall u_h \in V_j^h, \end{array}$$

where the constant $C \geq 0$ depends only on the uniformity parameter $C_U(\mathcal{C}\Omega_i)$ of the complement of Ω_i and the shape regularity of the finite elements and is otherwise independent of the finite element mesh sizes h_i and h_j and the diameters H_i and H_j.

We note that the inequalities of the following lemma are well known in the theory of iterative substructuring methods. Proofs for domains satisfying an interior cone condition are given in [3] and [4, Section 4.9] and a different proof is given in [30, Lemma 4.15]. For a proof that this inequality is sharp, see [5]. For a proof of the following lemma, which is for John domains, see [6].

Lemma 3.3 (Discrete Sobolev Inequality). *Let $\Omega \subset \mathbb{R}^2$ be a John domain with diameter H and $\bar{u}_\Omega$ denote the average of u over Ω. Then,*

$$\begin{array}{rcl} ||u - \bar{u}_\Omega||_{L^\infty(\Omega)}^2 & \leq & C(1 + log(H/h))|u|_{H^1(\Omega)}^2 \quad \text{and} \\ ||u||_{L^\infty(\Omega)}^2 & \leq & C(1 + log(H/h))||u||_{H^1(\Omega)}^2, \quad \forall u \in W^h(\Omega). \end{array}$$

The constant C depends only on the John parameter $C_J(\Omega)$ of Ω and the shape regularity of elements.

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^2$ be a John domain with diameter H. Then,*

$$||u - u(x_0)||_{L^2(\Omega)}^2 \leq CH^2(1 + log(H/h))|u|_{H^1(\Omega)}^2, \quad \forall x_0 \in \Omega. \tag{3.10}$$

Proof. We note that

$$||u - u(x_0)||_{L^2(\Omega)}^2 \leq 2||u - \bar{u}_\Omega||_{L^2(\Omega)}^2 + 2||\bar{u}_\Omega - u(x_0)||_{L^2(\Omega)}^2 \tag{3.11}$$

The first term on the right hand side of (3.11) can be estimated by an elementary Poincaré inequality, which holds for John domains [6, LEMMA 2.2]:

$$||u - \bar{u}_\Omega||_{L^2(\Omega)}^2 \leq CH^2|u|_{H^1(\Omega)}^2, \tag{3.12}$$

where C is a constant independent of the size of Ω. As for the second term on the right hand side,

$$\begin{array}{rl} & ||\bar{u}_\Omega - u(x_0)||_{L^2(\Omega)}^2 \\ \leq & |\Omega|||u - \bar{u}_\Omega||_{L^\infty(\Omega)}^2 \\ \leq & C|\Omega|(1 + log(H/h))|u|_{H^1(\Omega)}^2, \end{array} \tag{3.13}$$

where the second inequality follows from the discrete Sobolev inequality in Lemma 3.3. Combining (3.11),(3.12) and (3.13), we obtain (3.10). ☐

We need another tool to estimate energies of edge contributions in the two-dimensional case. For a proof of the following lemma for John domains, see [21, Lemma 4.4]. See [8] for the first proof of the same lemma for regular subdomains in two dimensions.

Lemma 3.5 (Edge Lemma)**.** *Let $\Omega_i \subset \mathbb{R}^2$ be a John domain, $\mathcal{E}^{ij} \subset \partial\Omega_i$ be an edge, and $\theta_{\mathcal{E}^{ij}} \in W^h(\Omega_i)$ be a finite element function which equals 1 at all nodes of $\mathcal{E}^{ij}$, vanishes at the other nodes on $\partial\Omega_i$, and is discrete harmonic in Ω_i. Then,*

$$|\mathcal{H}(\theta_{\mathcal{E}^{ij}}u)|^2_{H^1(\Omega_i)} \leq C(1+log(H/h))^2||u||^2_{H^1(\Omega_i)}, \quad \forall u \in W^h(\Omega_i), \tag{3.14}$$

$$|\theta_{\mathcal{E}^{ij}}|^2_{H^1(\Omega_i)} \leq C(1+log(H/h)) \tag{3.15}$$

and

$$||\theta_{\mathcal{E}^{ij}}||^2_{L^2(\Omega_i)} \leq CH^2(1+log(H/h)). \tag{3.16}$$

Here, C depends only on the John parameter $C_J(\Omega_i)$ of Ω_i and the shape regularity of the finite elements. The logarithmic factor of (3.16) can be removed for P_1 elements if all angles of the triangulation are acute.

We also note that the bound of Lemma 3.5 is independent of the length of the edge $\mathcal{E}^{ij}$.

We would need similar face and edge lemmas to advance the theory for the three-dimensional case. Theory for irregular subdomains in three dimensions is not complete at this point. However, such bounds have been established for regular subdomains; see [30, Chapter 4].

3.3 Technical Tools - part II

In this subsection, we present lemmas that are specific to the study of the FETI-FETI method.
We need a Poincaré-type inequality to treat the energy terms coming from the global interface between different bodies. We present such a lemma for the two-dimensional case. In the following lemma, which is an adaptation of [4, Lemma 10.6.6], we assume that we have Lipschitz subdomains.

Lemma 3.6. *Let $v \in \widetilde{W}^{(i)}$. Then*

$$||v||^2_{L^2(\Omega_i)} \leq C\left(H_b^2(1+log(H_s/h))^2\sum_j |v|^2_{H^1(\Omega_{i,j})} + \frac{1}{H_b^2}\left|\int_{\Omega_i} vdx\right|^2\right), \tag{3.17}$$

and

$$||v||^2_{L^2(\Omega_i)} \leq C\left(H_b^2(1+log(H_s/h))^2\sum_j |v|^2_{H^1(\Omega_{i,j})} + \left|\int_{\partial\Omega_i\cap\partial\Omega_D} vds\right|^2\right). \tag{3.18}$$

Proof. We follow the idea of [4, Lemma 10.6.6]. Let c be piecewise constant in each subdomain of Ω_i, i.e., $c(x) = c_{i,j}, \forall x \in \Omega_{i,j}, j = 1, \cdots, N_i$. We define a function $Ec \in H^1(\Omega_i)$ as follows. On the local interface of Ω_i, Ec is defined as the average of c, i.e.,

$$Ec(x) = \frac{\sum_{j\in\mathcal{N}_x} c_{i,j}(x)}{|\mathcal{N}_x|}, \quad x \in \Gamma^{(i)}_{loc,h},$$

where $\mathcal{N}_x$ is the set of indices of the subdomains of Ω_i with x on their boundaries. Also, we set $Ec|_{\partial\Omega_i} = c|_{\partial\Omega_i}$

With $Ec|_{\partial\Omega_{i,j}}, j = 1, \cdots, N_i$ given as above, Ec is defined to be discrete harmonic in each subdomain of Ω_i, i.e.,

$$|Ec|^2_{H^1(\Omega_{i,j})} = \min_{u|_{\partial\Omega_{i,j}}=Ec|_{\partial\Omega_{i,j}}} |u|^2_{H^1(\Omega_{i,j})}, \quad \forall j = 1, \cdots, N_i.$$

We have

$$||c||^2_{L^2(\Omega_i)}$$

$$\begin{aligned} &\leq 2||c - Ec||^2_{L^2(\Omega_i)} + 2||Ec||^2_{L^2(\Omega_i)} \\ &\leq 2\sum_j ||c - Ec||^2_{L^2(\Omega_{i,j})} + C\left(H_b^2|Ec|^2_{H^1(\Omega_i)} + \frac{1}{H_b^2}\left|\int_{\Omega_i} Ecdx\right|^2\right), \end{aligned} \tag{3.19}$$

where the second inequality follows from a standard Poincaré inequality with scaling. We estimate the first term of (3.19):

$$\begin{aligned} &\sum_j ||c - Ec||^2_{L^2(\Omega_{i,j})} \\ \leq\ & C\sum_j \left(H_s^2|c - Ec|^2_{H^1(\Omega_{i,j})} + H_s||c - Ec||^2_{L^2(\partial\Omega_{i,j})}\right), \end{aligned} \tag{3.20}$$

where the inequality is a Friedrichs inequality with scaling. We note that $c - Ec$ is constant on each edge of $\Omega_{i,j}, j = 1, \cdots, N_i$, and its values will often differ between different edges and vertices. Therefore we can write

$$(c - Ec)(x) = \left(\sum_{\mathcal{E}\subset\partial\Omega_{i,j}} \theta_{\mathcal{E}}(c - Ec) + \sum_{\mathcal{V}\subset\partial\Omega_{i,j}} \theta_{\mathcal{V}}(c - Ec)\right)(x), \quad \forall x \in \partial\Omega_{i,j,h}.$$

We note that the characteristic function $\theta_{\mathcal{E}}$ has already been defined in Lemma 3.5. $\theta_{\mathcal{V}}$ is defined analogously; it equals 1 at $\mathcal{V}$, vanishes at all other nodes on $\partial\Omega_{i,j}$ and is discrete harmonic in $\Omega_{i,j}$. Noting that $c - Ec$ is discrete harmonic in $\Omega_{i,j}$, we have

$$\begin{aligned} &|c - Ec|^2_{H^1(\Omega_{i,j})} \\ \leq\ & \left|\sum_{\mathcal{E}\subset\partial\Omega_{i,j}} \theta_{\mathcal{E}}(c - Ec) + \sum_{\mathcal{V}\subset\partial\Omega_{i,j}} \theta_{\mathcal{V}}(c - Ec)\right|^2_{H^1(\Omega_{i,j})} \\ \leq\ & C\left(\sum_{\mathcal{E}\subset\partial\Omega_{i,j}} |\theta_{\mathcal{E}}(c - Ec)|^2_{H^1(\Omega_{i,j})} + \sum_{\mathcal{V}\subset\partial\Omega_{i,j}} |\theta_{\mathcal{V}}(c - Ec)|^2_{H^1(\Omega_{i,j})}\right). \end{aligned} \tag{3.21}$$

We define $[[c]]_e$, the *jump* of the function c across the edge e; if e is shared by $\partial\Omega_{i,j}$ and $\partial\Omega_{i,k}$, $c|_{\Omega_{i,j}} = c_j, c|_{\Omega_{i,k}} = c_k$, then $[[c]]_e := |c_j - c_k|$. Letting $\mathcal{E}^i(\Omega_i)$ denote the set of the interior edges of Ω_i, we have

$$\begin{aligned} &\sum_j \sum_{\mathcal{E}\subset\partial\Omega_{i,j}} |\theta_{\mathcal{E}}(c - Ec)|^2_{H^1(\Omega_{i,j})} \\ \leq\ & \sum_j \sum_{\mathcal{E}\subset\partial\Omega_{i,j}} ||c - Ec||^2_{L^\infty(\mathcal{E})}|\theta_{\mathcal{E}}|^2_{H^1(\Omega_{i,j})} \\ \leq\ & C(1 + log(H_s/h))\sum_j \sum_{\mathcal{E}\subset\partial\Omega_{i,j}} ||c - Ec||^2_{L^\infty(\mathcal{E})} \\ \leq\ & C(1 + log(H_s/h))\sum_{e\in\mathcal{E}^i(\Omega_i)} [[c]]_e^2 \\ \leq\ & C(1 + log(H_s/h))\sum_{e\in\mathcal{E}^i(\Omega_i)} \frac{1}{|e|}||[[c]]_e||^2_{L^2(e)}, \end{aligned} \tag{3.22}$$

where the second inequality follows from Lemma 3.5. Using the fact that $|\theta_{\mathcal{V}}|^2_{H^1(\Omega_{i,j})} = O(1)$, we have

$$\begin{aligned} &\sum_j \sum_{\mathcal{V}\subset\partial\Omega_{i,j}} |\theta_{\mathcal{V}}(c - Ec)|^2_{H^1(\Omega_{i,j})} \\ \leq\ & C\sum_j \sum_{\mathcal{V}\subset\partial\Omega_{i,j}} |(c - Ec)|_{\mathcal{V}}|^2 \\ \leq\ & C\sum_{e\in\mathcal{E}^i(\Omega_i)} [[c]]_e^2 \end{aligned} \tag{3.23}$$

We note that (3.23) can be absorbed into (3.22). Combining (3.21), (3.22), and (3.23), we obtain

$$\begin{aligned}\sum_j |Ec|^2_{H^1(\Omega_{i,j})} &= \sum_j |c - Ec|^2_{H^1(\Omega_{i,j})} \\ &\leq C(1 + log(H_s/h)) \sum_{e \in \mathcal{E}^i(\Omega_i)} \frac{1}{|e|} ||[[c]]_e||^2_{L^2(e)}, \qquad (3.24)\end{aligned}$$

where the equality follows from the fact that c is a constant in $\Omega_{i,j}, \forall j$. The second term of (3.20) is handled similarly:

$$\begin{aligned}&||c - Ec||^2_{L^2(\partial\Omega_{i,j})} \\ = \ & \left|\left| \sum_{\mathcal{E} \subset \partial\Omega_{i,j}} \theta_{\mathcal{E}}(c - Ec) + \sum_{\mathcal{V} \subset \partial\Omega_{i,j}} \theta_{\mathcal{V}}(c - Ec) \right|\right|^2_{L^2(\partial\Omega_{i,j})} \\ \leq \ & C \left(\sum_{\mathcal{E} \subset \partial\Omega_{i,j}} ||\theta_{\mathcal{E}}(c - Ec)||^2_{L^2(\partial\Omega_{i,j})} + \sum_{\mathcal{V} \subset \partial\Omega_{i,j}} ||\theta_{\mathcal{V}}(c - Ec)||^2_{L^2(\partial\Omega_{i,j})} \right). \qquad (3.25)\end{aligned}$$

Then

$$\begin{aligned}&\sum_j \sum_{\mathcal{E} \subset \partial\Omega_{i,j}} ||\theta_{\mathcal{E}}(c - Ec)||^2_{L^2(\partial\Omega_{i,j})} \\ \leq \ & \sum_j \sum_{\mathcal{E} \subset \partial\Omega_{i,j}} ||c - Ec||^2_{L^\infty(\mathcal{E})} ||\theta_{\mathcal{E}}||^2_{L^2(\partial\Omega_{i,j})} \\ \leq \ & CH_s \sum_j \sum_{\mathcal{E} \subset \partial\Omega_{i,j}} ||c - Ec||^2_{L^\infty(\mathcal{E})} \\ \leq \ & CH_s \sum_{e \in \mathcal{E}^i(\Omega_i)} [[c]]^2_e \\ \leq \ & CH_s \sum_{e \in \mathcal{E}^i(\Omega_i)} \frac{1}{|e|} ||[[c]]_e||^2_{L^2(e)}. \qquad (3.26)\end{aligned}$$

Also,

$$\begin{aligned}&\sum_j \sum_{\mathcal{V} \subset \partial\Omega_{i,j}} ||\theta_{\mathcal{V}}(c - Ec)||^2_{L^2(\partial\Omega_{i,j})} \\ \leq \ & Ch \sum_j \sum_{\mathcal{V} \subset \partial\Omega_{i,j}} |(c - Ec)|_{\mathcal{V}}|^2 \\ \leq \ & Ch \sum_{e \in \mathcal{E}^i(\Omega_i)} [[c]]^2_e, \qquad (3.27)\end{aligned}$$

which can be absorbed into (3.26) as before. Combining (3.19) - (3.27), we have

$$\begin{aligned}&||c||^2_{L^2(\Omega_i)} \\ \leq \ & CH_b^2(1 + log(H_s/h)) \sum_{e \in \mathcal{E}^i(\Omega_i)} \frac{1}{|e|} ||[[c]]_e||^2_{l^2(e)} + C\frac{1}{H_b^2} \left| \int_{\Omega_i} Ecdx \right|^2 \\ \leq \ & CH_b^2 H_s^{-1}(1 + log(H_s/h)) \sum_{e \in \mathcal{E}^i(\Omega_i)} ||[[c]]_e||^2_{L^2(e)} + C\frac{1}{H_b^2} \left| \int_{\Omega_i} Ecdx \right|^2. \qquad (3.28)\end{aligned}$$

The rest of the proof of (3.17) is similar to the proof of [4, Lemma 10.6.7]; let $v \in \widetilde{W}^{(i)}$ and $\bar{v}$ be defined by

$$\bar{v}(x) = \frac{1}{|\Omega_{i,j}|} \int_{\Omega_{i,j}} vdx, \quad \forall x \in \Omega_{i,j}, \quad j = 1, \cdots, N_i.$$

Then,

$$\begin{aligned}
& ||v||^2_{L^2(\Omega_i)} \\
\leq\ & 2||v-\bar{v}||^2_{L^2(\Omega_i)} + 2||\bar{v}||^2_{L^2(\Omega_i)} \\
\leq\ & 2\sum_j ||v-\bar{v}||^2_{L^2(\Omega_{i,j})} + \\
& C\Big(H_b^2 H_s^{-1}(1+log(H_s/h)) \sum_{e\in\mathcal{E}^i(\Omega_i)} ||[[\bar{v}]]_e||^2_{L^2(e)} + \frac{1}{H_b^2}\left|\int_{\Omega_i}\bar{v}dx\right|^2\Big) \\
\leq\ & C\Big(\sum_j H_s^2|v|^2_{H^1(\Omega_{i,j})} + H_b^2 H_s^{-1}(1+log(H_s/h)) \sum_{e\in\mathcal{E}^i(\Omega_i)} ||[[\bar{v}]]_e||^2_{L^2(e)} + \frac{1}{H_b^2}\left|\int_{\Omega_i} vdx\right|^2\Big),
\end{aligned} \tag{3.29}$$

where the second inequality follows from (3.28). We now estimate the second to the last term. We first note that

$$\sum_{e\in\mathcal{E}^i(\Omega_i)} ||[[\bar{v}]]_e||^2_{L^2(e)} \leq 2\sum_{e\in\mathcal{E}^i(\Omega_i)} ||[[v-\bar{v}]]_e||^2_{L^2(e)} + 2\sum_{e\in\mathcal{E}^i(\Omega_i)} ||[[v]]_e||^2_{L^2(e)},$$

and estimate these two terms separately.

$$\begin{aligned}
& \sum_{e\in\mathcal{E}^i(\Omega_i)} ||[[v-\bar{v}]]_e||^2_{L^2(e)} \\
\leq\ & C\sum_j ||v-\bar{v}||^2_{L^2(\partial\Omega_{i,j})} \\
\leq\ & C\sum_j \Big(H_s|v-\bar{v}|^2_{H^1(\Omega_{i,j})} + H_s^{-1}||v-\bar{v}||^2_{L^2(\Omega_{i,j})}\Big) \\
\leq\ & CH_s\sum_j |v-\bar{v}|^2_{H^1(\Omega_{i,j})},
\end{aligned} \tag{3.30}$$

where the second inequality follows from a trace theorem and the third from a Poincaré inequality with scaling.

Assuming e is shared by $\Omega_{i,j}$ and $\Omega_{i,k}$ and $\mathcal{V}$ is a vertex of e, we have

$$\begin{aligned}
& ||[[v]]_e||^2_{L^2(e)} \\
=\ & ||v_{i,j}-v_{i,k}||^2_{L^2(e)} \\
\leq\ & 2\left(||v_{i,j}-v_{i,j}(\mathcal{V})||^2_{L^2(e)} + ||v_{i,k}-v_{i,k}(\mathcal{V})||^2_{L^2(e)}\right) \\
\leq\ & C\left(H_s||v_{i,j}-v_{i,j}(\mathcal{V})||^2_{L^\infty(\Omega_{i,j})} + H_s||v_{i,k}-v_{i,k}(\mathcal{V})||^2_{L^\infty(\Omega_{i,k})}\right) \\
\leq\ & CH_s(1+log(H_s/h))\left(|v|^2_{H^1(\Omega_{i,j})} + |v|^2_{H^1(\Omega_{i,k})}\right),
\end{aligned} \tag{3.31}$$

where the last inequality follows from [30, Lemma 4.15]. Combining (3.29), (3.30), and (3.31), we obtain (3.17).

Similarly, we have

$$\begin{aligned}
& ||v||^2_{L^2(\Omega_i)} \\
\leq\ & 2||v-\bar{v}||^2_{L^2(\Omega_i)} + 2||\bar{v}||^2_{L^2(\Omega_i)} \\
\leq\ & 2\sum_j ||v-\bar{v}||^2_{L^2(\Omega_{i,j})} + CH_b^2H_s^{-1}(1+log(H_s/h))\Big(\sum_{e\in\mathcal{E}^i(\Omega_i)} ||[[\bar{v}]]_e||^2_{L^2(\Omega_i)} \\
& + \left|\int_{\partial\Omega_i\cap\partial\Omega_D}\bar{v}ds\right|^2\Big) \\
\leq\ & 2\sum_j ||v-\bar{v}||^2_{L^2(\Omega_{i,j})} + CH_b^2H_s^{-1}(1+log(H_s/h))\Big(\sum_{e\in\mathcal{E}^i(\Omega_i)} ||[[\bar{v}]]_e||^2_{L^2(\Omega_i)} +
\end{aligned}$$

$$\left|\int_{\partial\Omega_i\cap\partial\Omega_D} vds\right|^2 + \left|\int_{\partial\Omega_i\cap\partial\Omega_D} (v-\bar{v})ds\right|^2\Bigg). \tag{3.32}$$

Letting $\mathcal{E}^b(\Omega_i)$ denote the set of exterior subdomain edges of Ω_i, we have

$$\begin{aligned}
& \left|\int_{\partial\Omega_i\cap\partial\Omega_D} (v-\bar{v})dx\right|^2 \\
\leq \; & |\partial\Omega_i\cap\partial\Omega_D| \int_{\partial\Omega_i\cap\partial\Omega_D} (v-\bar{v})^2 ds \\
\leq \; & |\partial\Omega_i\cap\partial\Omega_D| \sum_{e\in\mathcal{E}^b(\Omega_i)} ||v-\bar{v}||^2_{L^2(e)} \\
\leq \; & |\partial\Omega_i\cap\partial\Omega_D| \sum_j ||v-\bar{v}||^2_{L^2(\partial\Omega_{i,j})} \\
\leq \; & C|\partial\Omega_i\cap\partial\Omega_D| \sum_j H_s|v|^2_{H^1(\Omega_{i,j})} \\
\leq \; & CH_bH_s \sum_j |v|^2_{H^1(\Omega_{i,j})},
\end{aligned} \tag{3.33}$$

where the fourth inequality follows from a trace theorem and a Poincaré's inequality with scaling. Combining (3.32), (3.33), (3.30), and (3.31), we obtain (3.18). □

Remark 1. *We note that* $|\int_{\Omega_i} vdx|^2$ *and* $|\int_{\partial\Omega_i\cap\partial\Omega_D} vds|^2$ *of (3.17) and (3.18), respectively, can be replaced by* $\sum_{k=1}^L |f_k(v)|^2$ *(with a proper scaling factor), where* $f_k, k=1,\cdots,L, L\geq 1$ *are functionals (not necessarily linear) in* $H^1(\Omega_i)$*, such that, if* v *is constant in* Ω*,*

$$\sum_{k=1}^L |f_k(v)|^2 = 0 \Leftrightarrow v=0;$$

see [29, Lemma 2.7.1], [30, Appendix A.4]. Indeed, this fact will be used in the proof of Lemma 3.9.

We now provide a trace theorem for functions in $\widetilde{W}^{(i)}$, for $\Omega_i \in \mathbb{R}^2$ with Lipschitz subdomains, $\Omega_{i,j}, j=1,\cdots,N_i$.

Lemma 3.7. *Let* $u\in\widetilde{W}^{(i)}$*. We then have*

$$||u||^2_{L^2(\partial\Omega_i)} \leq C\left(H_b(1+log(H_s/h))^2\sum_{j=1}^{N_i}|u|^2_{H^1(\Omega_{i,j})} + \frac{1}{H_b}||u||^2_{L^2(\Omega_i)}\right). \tag{3.34}$$

Proof. We adopt the framework of the proof of Lemma 3.6. Let c be piecewise constant in each subdomain of Ω_i, i.e., $c(x)=c_{i,j}, \forall x\in\Omega_{i,j}, j=1,\cdots,N_i$. We define the function Ec as before, i.e., Ec is defined to be the average of c on $\Gamma^{(i)}_{loc}, Ec|_{\partial\Omega_i} = c|_{\partial\Omega_i}$, and discrete harmonic in each subdomain $\Omega_{i,j}, j=1,\cdots,N_i$. Then,

$$\begin{aligned}
& ||c||^2_{L^2(\partial\Omega_i)} \\
= \; & ||Ec||^2_{L^2(\partial\Omega_i)} \\
\leq \; & C\left(H_b|Ec|^2_{H^1(\Omega_i)} + \frac{1}{H_b}||Ec||^2_{L^2(\Omega_i)}\right) \\
\leq \; & C\left(H_b\sum_j|Ec|^2_{H^1(\Omega_{i,j})} + \frac{1}{H_b}||c||^2_{L^2(\Omega_i)} + \frac{1}{H_b}||c-Ec||^2_{L^2(\Omega_i)}\right) \\
\leq \; & C\left(H_bH_s^{-1}(1+log(H_s/h))\sum_{e\in\mathcal{E}^i(\Omega_i)}||[[c]]_e||^2_{L^2(e)} + \frac{1}{H_b}||c||^2_{L^2(\Omega_i)}\right),
\end{aligned} \tag{3.35}$$

where the first and the third inequalities follow from a trace theorem with scaling and (3.20) - (3.27), respectively. Now let v satisfy the assumption of the lemma and $\bar{v}$ be a piecewise constant function which is the average of v in each subdomain of Ω_i. Then,

$$\begin{aligned}
& ||v||^2_{L^2(\partial\Omega_i)} \\
\leq \; & 2||v-\bar{v}||^2_{L^2(\partial\Omega_i)} + 2||\bar{v}||^2_{L^2(\partial\Omega_i)} \\
\leq \; & C\Big(H_b H_s^{-1}(1+log(H_s/h)) \sum_{e\in\mathcal{E}^i(\Omega_i)} ||[[\bar{v}]]_e||^2_{L^2(e)} + \frac{1}{H_b}||v||^2_{L^2(\Omega_i)} + \\
& ||v-\bar{v}||^2_{L^2(\partial\Omega_i)}\Big). \qquad (3.36)
\end{aligned}$$

Using a trace theorem and a Poincaré's inequality with scaling at the subdomain level, we obtain

$$\begin{aligned}
& ||v-\bar{v}||^2_{L^2(\partial\Omega_i)} \\
\leq \; & \sum_j ||v-\bar{v}||^2_{L^2(\partial\Omega_{i,j})} \\
\leq \; & C\sum_j \left(H_s|v-\bar{v}|^2_{H^1(\Omega_{i,j})} + \frac{1}{H_s}||v-\bar{v}||^2_{L^2(\Omega_{i,j})}\right) \\
\leq \; & C\sum_j H_s|v|^2_{H^1(\Omega_{i,j})}. \qquad (3.37)
\end{aligned}$$

Bounding $\sum_{e\in\mathcal{E}^i(\Omega_i)} ||[[\bar{v}]]_e||^2_{L^2(e)}$ as in (3.30) and (3.31), we have

$$\begin{aligned}
& ||v||^2_{L^2(\partial\Omega_i)} \\
\leq \; & C\left(H_b(1+log(H_s/h))^2 \sum_j |v|^2_{H^1(\Omega_{i,j})} + \frac{1}{H_b}||v||^2_{L^2(\Omega_i)}\right). \qquad (3.38)
\end{aligned}$$

□

Before we complete this section, we comment on the analysis of linear elasticity problems in two dimensions. Korn inequality is essential in any such analysis and here we present a version for Jones (i.e., uniform) domains. For a proof, see Durán and Muschietti [9].

Lemma 3.8 (Korn inequality for uniform domains)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain. Then, there exists C, which depends only on the Jones parameter $C_U(\Omega)$ and the dimension n, such that*

$$|\mathbf{u}|_{H^1(\Omega)} \leq C\sum_{i,j} ||\epsilon(\mathbf{u})_{ij}||^2_{L^2(\Omega)}$$

for all $\mathbf{u} \in \{\mathbf{u} \in \mathbf{H^1}(\Omega) : \int_\Omega (\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i})dx = 0, i,j = 1,\cdots,n\}$.

Using this Korn inequality, the analysis for a two-dimensional compressible linear elasticity problem with Lipschitz domains can be carried out without much difficulty. We would also need to consider different primal constraints on each body, especially for the case of three-dimensional problem; see [23] for details.

3.4 Convergence bound for the FETI-FETI method

In this section, we assume that our two-dimensional subdomains have Lipschitz boundaries. Recall that $\widetilde{P}^k_D := \widetilde{B}^T_{D,\Gamma_c} Z^k_\Gamma \widetilde{B}_{\Gamma,c}$, and thus the operator $\widetilde{P}^k_D$ changes in each step of an active set method. We prove

Lemma 3.9. *For any $w \in range(\widetilde{S}_c)$, we have*

$$|\widetilde{P}^k_D w|^2_{\widetilde{S}_c} \leq C\frac{H_b}{H_s}(1+log(H_s/h))^2 |w|^2_{\widetilde{S}_c},$$

where $C > 0$ is a constant independent of H_b, H_s, and h.

Remark 2. *We note that* $w \in range(\widetilde{S}_c)$ *is equivalent to* $l_i(w_i) := \mathbf{1}_i^T w_i = 0$*, where* i *indicates the index of any body which does not intersect* $\partial\Omega_D$ *and* $\mathbf{1}_i \in \widetilde{W}_\Gamma^{(i)}$ *denotes a constant vector of ones. As mentioned in Remark 1, we can use* $l_i(v)$ *in place of* $|\int_{\Omega_i} v dx|^2$ *or* $|\int_{\partial\Omega_i \cap \partial\Omega_D} v ds|^2$*, with a proper scaling factor, in Lemma 3.6.*

We first make some observations. We obtain the following formulae by modifying [30, (6.42)]:

$$(\widetilde{P}_D^k w(x))_{i,j} = \sum_{s \in \mathcal{N}_{x,loc}^{(i)}} \delta_{i,s}^\dagger(x)(w_{i,j}(x) - w_{i,s}(x)), \quad \text{if} \quad x \in \partial\Omega_{i,j} \cap \Gamma_{loc}^{(i)}, \tag{3.39}$$

$$(\widetilde{P}_D^k w(x))_i = \sum_{k \in \mathcal{N}_{x,gl}} \delta_k^\dagger(x)(w_i(x) - w_k(x)), \quad \text{if} \quad x \in \partial\Omega_i \cap \Gamma_{gl}^k. \tag{3.40}$$

Also, recalling that $\widetilde{P}_D^k := \widetilde{B}_{D,\Gamma_c}^T Z_\Gamma^k \widetilde{B}_{\Gamma_c}$,

$$(\widetilde{P}_D^k w(x))_i = 0, \quad \text{if} \quad x \in \partial\Omega_i \cap \overline{\Gamma_{gl}} \setminus \Gamma_{gl}^k. \tag{3.41}$$

The equation (3.41) is due to the fact that the entries of the square matrix Z_Γ^k that correspond to the nodes not belonging to the current active area Γ_{gl}^k are zero. In (3.40) and (3.41), we do not specify subdomain indices since we are considering nodes belonging to only one subdomain. However, in the following discussion, we sometimes do specify the relevant subdomain indices when necessary.

We follow the proof of [30, Lemma 6.3] very closely. However, one of the main differences between the proof there and the proof we present here is that we mainly work with H^1- seminorms instead of $H^{1/2}$- seminorms.

We will denote the discrete harmonic extension operator by $\mathcal{H}$; $\mathcal{H}(v)$ is the minimal energy extension of the finite element function v, defined on the subdomain interface, into the interior of the subdomain.

Proof. Recalling the definition of $\widetilde{S}_\Gamma$ from subsection 3.1, we have

$$|\widetilde{P}_D^k w|_{\widetilde{S}_c}^2 = \sum_{i=1}^N |(\widetilde{P}_D^k w)_i|_{\widetilde{S}_\Gamma^{(i)}}^2, \quad |w|_{\widetilde{S}_c}^2 = \sum_{i=1}^N |w_i|_{\widetilde{S}_\Gamma^{(i)}}^2.$$

Therefore suffices to show that

$$|(\widetilde{P}_D^k w)_i|_{\widetilde{S}_\Gamma^{(i)}}^2 \le C\frac{H_b}{H_s}(1 + log(H_s/h))^2 |w_i|_{\widetilde{S}_\Gamma^{(i)}}^2, \quad i = 1, 2, \cdots, N.$$

Furthermore,

$$|(\widetilde{P}_D^k w)_i|_{\widetilde{S}_\Gamma^{(i)}}^2 = \sum_{j=1}^{N_i} |(\widetilde{P}_D^k w)_{i,j}|_{S^{i,j}}^2,$$

where $(\widetilde{P}_D^k w)_{i,j}$ is the restriction of $(\widetilde{P}_D^k w)_i \in \widetilde{W}_\Gamma^{(i)}$ to $W_\Gamma^{(i,j)}$. Thus it suffices to examine each $|(\widetilde{P}_D^k w)_{i,j}|_{S^{(i,j)}}$ separately. For notational simplicity, let $v_{i,j}(x) := (\widetilde{P}_D^k w)_{i,j}$. We can see that the coefficients in (3.39) and (3.40) are constant on each individual edge while their values will differ between different edges. Also, $(\widetilde{P}_D^k w(x))_{i,j} = 0$ for a vertex node $x \in \Gamma_{loc}^{(i)}$, since we are imposing continuity at all vertices of the same body. Therefore it makes sense to write $v_{i,j}$ as a sum of functions each of which vanishes at all interface nodes outside a certain edge or a vertex. We can accomplish this by using characteristic finite element functions for individual edges and vertices; such characteristic functions for an edge and a vertex, $\theta_\mathcal{E}$ and $\theta_\mathcal{V}$, have already been introduced in Lemma 3.5 and Lemma 3.6, respectively. Construction of these finite element functions and the proof of their characteristics for three-dimensional problems can be found in [30, Chapter 4]. Construction of $\theta_\mathcal{E}$ and $\theta_\mathcal{V}$ for two-dimensional problems is analogous and here we just present their characteristics without any proofs.

Two-dimensional Case

Using the partition of unity,

$$v_{i,j}(x) = \sum_{\mathcal{E} \subset \partial\Omega_{i,j} \cap \Gamma_{gl}^k} I^h(\theta_\mathcal{E} v_{i,j}(x)) + \sum_{\mathcal{E} \subset \partial\Omega_{i,j} \cap \Gamma_{loc}^{(i)}} I^h(\theta_\mathcal{E} v_{i,j}(x)) + \sum_{\mathcal{V} \subset \partial\Omega_{i,j} \cap \Gamma_{gl}^k} I^h(\theta_\mathcal{V} v_{i,j}(x)), \quad \forall x \in \partial\Omega_{i,j,h},$$

where I^h is the standard nodal interpolation operator. We first consider the terms for the edges on the global interface Γ^k_{gl}.

Edge Terms - Global Interface

Suppose $\mathcal{E}$ is shared by $\partial\Omega_{i,j}$ and $\partial\Omega_{k,l}$, where $i \neq k$. Then,

$$I^h(\theta_{\mathcal{E}} v_{i,j}) = I^h(\theta_{\mathcal{E}} \delta_k^\dagger(\mathcal{E})(w_{i,j} - w_{k,l})),$$

where $\delta_k^\dagger(\mathcal{E})$ is the constant value of $\delta_k^\dagger(x)$ on the edge $\mathcal{E}$. We have

$$|I^h(\theta_{\mathcal{E}} v_{i,j})|^2_{S^{(i,j)}} = \rho_{i,j}|\mathcal{H}(\theta_{\mathcal{E}} v_{i,j})|^2_{H^1(\Omega_{i,j})}, \tag{3.42}$$

using the equivalence of norms of discrete harmonic extensions and traces on the subdomain boundaries; see [30, Section 4.4]. We then have,

$$\begin{aligned}
& \rho_{i,j}|\mathcal{H}(\theta_{\mathcal{E}} v_{i,j})|^2_{H^1(\Omega_{i,j})} \\
= \quad & \rho_{i,j}|\mathcal{H}(\theta_{\mathcal{E}}(\delta_k^\dagger(\mathcal{E})(w_{i,j} - w_{k,l})))|^2_{H^1(\Omega_{i,j})} \\
\leq \quad & 2\rho_{i,j}\delta_k^\dagger(\mathcal{E})^2(|\mathcal{H}(\theta_{\mathcal{E}}(w_{i,j}))|^2_{H^1(\Omega_{i,j})} + |\mathcal{H}(\theta_{\mathcal{E}}(w_{k,l}))|^2_{H^1(\Omega_{i,j})}) \\
\leq \quad & 2\min(\rho_{i,j}, \rho_{k,l})(|\mathcal{H}(\theta_{\mathcal{E}}(w_{i,j}))|^2_{H^1(\Omega_{i,j})} + |\mathcal{H}(\theta_{\mathcal{E}}(w_{k,l}))|^2_{H^1(\Omega_{i,j})}).
\end{aligned}$$

Here, the second inequality follows from [23, Lemma 8.4]; see also [30, (6.19)]. We treat the first term using Lemma 3.5:

$$\begin{aligned}
& 2\min(\rho_{i,j}, \rho_{k,l})|\mathcal{H}(\theta_{\mathcal{E}}(w_{i,j}))|^2_{H^1(\Omega_{i,j})} \\
\leq \quad & 2\rho_{i,j}(1 + log(H_s/h))^2||\mathcal{H}(w_{i,j})||^2_{H^1(\Omega_{i,j})}
\end{aligned}$$

For the second term, we also need the extension lemma:

$$\begin{aligned}
& 2\min(\rho_{i,j}, \rho_{k,l})|\mathcal{H}(\theta_{\mathcal{E}}(w_{k,l}))|^2_{H^1(\Omega_{i,j})} \\
\leq \quad & 2C\rho_{k,l}|E^h_{kl,ij}(\mathcal{H}(\theta_{\mathcal{E}}(w_{k,l})))|^2_{H^1(\Omega_{i,j})} \\
\leq \quad & 2C\rho_{k,l}||\mathcal{H}(\theta_{\mathcal{E}}(w_{k,l}))||^2_{H^1(\Omega_{k,l})} \\
\leq \quad & 2C\rho_{k,l}(1 + log(H_s/h))^2||\mathcal{H}(w_{k,l})||^2_{H^1(\Omega_{k,l})}.
\end{aligned}$$

We sum (3.42) over j, k and l, which indicate the indices of all subdomains of Ω_i intersecting the global interface ${\Gamma_{gl}}^k$, the indices of all bodies sharing their boundaries with Ω_i and the collection of the indices of the subdomains of Ω_k sharing their boundaries with those of Ω_i, respectively:

$$\begin{aligned}
& \sum\nolimits_j |I^h(\theta_{\mathcal{E}} v)|^2_{S^{(i,j)}} \\
\leq \quad & C(1 + log(H_s/h))^2 \left(\sum\nolimits_j \rho_{i,j}||\mathcal{H}(w_{i,j})||^2_{H^1(\Omega_{i,j})} + \sum\nolimits_{k,l} \rho_{k,l}||\mathcal{H}(w_{k,l})||^2_{H^1(\Omega_{k,l})}\right) \\
= \quad & C(1 + log(H_s/h))^2 \left(\sum\nolimits_j \rho_{i,j}|\mathcal{H}(w_{i,j})|^2_{H^1(\Omega_{i,j})} + \sum\nolimits_{k,l} \rho_{k,l}|\mathcal{H}(w_{k,l})|^2_{H^1(\Omega_{k,l})}\right) \\
+ \quad & C(1 + log(H_s/h))^2 H_s^{-2}\left(\sum\nolimits_j \rho_{i,j}||\mathcal{H}(w_{i,j})||^2_{L^2(\Omega_{i,j})} + \sum\nolimits_{k,l} \rho_{k,l}||\mathcal{H}(w_{k,l})||^2_{L^2(\Omega_{k,l})}\right).
\end{aligned} \tag{3.43}$$

We control the L^2- terms of (3.43) using a similar argument as in [30, Lemma 3.10]. Using a Friedrichs inequality for each subdomain of Ω_i which intersects $\partial\Omega_i$, we get

$$\rho_{i,j}||\mathcal{H}(w_{i,j})||^2_{L^2(\Omega_{i,j})} \leq C\left(H_s^2\rho_{i,j}|\mathcal{H}(w_{i,j})|^2_{H^1(\Omega_{i,j})} + H_s\rho_{i,j}||w_{i,j}||^2_{L^2(\partial\Omega_{i,j}\cap\partial\Omega_i)}\right). \tag{3.44}$$

Summing over $\partial\Omega_i$, we get

$$\begin{aligned}
& \sum\nolimits_j \rho_{i,j}||\mathcal{H}(w_{i,j})||^2_{L^2(\Omega_{i,j})} \\
\leq \quad & C\left(H_s^2\sum\nolimits_j \rho_{i,j}|\mathcal{H}(w_{i,j})|^2_{H^1(\Omega_{i,j})} + H_s\rho_i||\mathcal{H}(w_i)||^2_{L^2(\partial\Omega_i)}\right)
\end{aligned}$$

$$\begin{aligned}
&\le C\Big(H_s^2\sum\nolimits_j\rho_{i,j}|\mathcal{H}(w_{i,j})|^2_{H^1(\Omega_{i,j})} + H_bH_s(1+log(H_s/h))^2\rho_i\sum_{s=1}^{N_i}|\mathcal{H}(w_{i,s})|^2_{H^1(\Omega_{i,s})}+\\
&\quad \frac{H_s}{H_b}\rho_i\sum_{s=1}^{N_i}||\mathcal{H}(w_{i,s})||^2_{L^2(\Omega_{i,s})}\Big)\\
&\le CH_bH_s(1+log(H_s/h))^2\rho_i\sum\nolimits_j|\mathcal{H}(w_{i,j})|^2_{H^1(\Omega_{i,j})}+\\
&\quad C\frac{H_s}{H_b}\rho_i\cdot H_b^2(1+log(H_s/h))^2\sum\nolimits_j|\mathcal{H}(w_{i,j})|^2_{H^1(\Omega_{i,j})}\\
&\le CH_bH_s(1+log(H_s/h))^2\rho_i\sum\nolimits_j|\mathcal{H}(w_{i,j})|^2_{H^1(\Omega_{i,j})}.
\end{aligned}$$

where the second inequality follows from Lemma 3.7 and the third from Lemma 3.6 and Remark 2. We also note that we repeatedly use (3.4). The last term of (3.43) can be handled similarly.

Edge Terms - Local Interface

Suppose $\mathcal{E}$ is shared by $\partial\Omega_{i,j}$ and $\partial\Omega_{i,s}$. Then,

$$I^h(\theta_\mathcal{E}v_{i,j}) = I^h(\theta_\mathcal{E}\delta^\dagger_{i,s}(\mathcal{E})(w_{i,j}-w_{i,s})),$$

where $\delta^\dagger_{i,s}(\mathcal{E})$ is the constant value of $\delta^\dagger_{i,s}(x)$ on the edge $\mathcal{E}$.

With $\bar{w}_{i,j} := \int_{\Omega_{i,j}} w_{i,j}dx/\int_{\Omega_{i,j}}1dx$ and $\bar{w}_{i,s} := \int_{\Omega_{i,s}} w_{i,s}dx/\int_{\Omega_{i,s}}1dx$, we have

$$\begin{aligned}
&|I^h(\theta_\mathcal{E}v_{i,j})|^2_{S^{(i,j)}}\\
=\ &\rho_{i,j}|\mathcal{H}(\theta_\mathcal{E}v_{i,j})|^2_{H^1(\Omega_{i,j})}\\
=\ &\rho_{i,j}|\mathcal{H}(\theta_\mathcal{E}\delta^\dagger_{i,s}(\mathcal{E})(w_{i,j}-w_{i,s}))|^2_{H^1(\Omega_{i,j})}\\
=\ &\rho_{i,j}|\mathcal{H}(\theta_\mathcal{E}\delta^\dagger_{i,s}(\mathcal{E})((w_{i,j}-\bar{w}_{i,j})-(w_{i,s}-\bar{w}_{i,s})+(\bar{w}_{i,j}-\bar{w}_{i,s})))|^2_{H^1(\Omega_{i,j})}\\
\le\ &3\rho_{i,j}|\mathcal{H}(\theta_\mathcal{E}\delta^\dagger_{i,s}(\mathcal{E})(w_{i,j}-\bar{w}_{i,j})|^2_{H^1(\Omega_{i,j})}\\
+\ &3\rho_{i,j}|\mathcal{H}(\theta_\mathcal{E}\delta^\dagger_{i,s}(\mathcal{E})(w_{i,s}-\bar{w}_{i,s})|^2_{H^1(\Omega_{i,j})}\\
+\ &3\rho_{i,j}|\theta_\mathcal{E}\delta^\dagger_{i,s}(\mathcal{E})(\bar{w}_{i,j}-\bar{w}_{i,s})|^2_{H^1(\Omega_{i,j})}
\end{aligned}\tag{3.45}$$

We can estimate the first term using Lemma 3.5, a Poincaré inequality and [23, Lemma 8.4]:

$$\begin{aligned}
&\rho_{i,j}|\mathcal{H}(\theta_\mathcal{E}\delta^\dagger_{i,s}(\mathcal{E})(w_{i,j}-\bar{w}_{i,j})|^2_{H^1(\Omega_{i,j})}\\
\le\ &C\rho_{i,j}\delta^\dagger_{i,s}(\mathcal{E})^2(1+log(H_s/h))^2||\mathcal{H}(w_{i,j})-\bar{w}_{i,j}||^2_{H^1(\Omega_{i,j})}\\
\le\ &C\rho_{i,j}(1+log(H_s/h))^2|\mathcal{H}(w_{i,j})|^2_{H^1(\Omega_{i,j})}.
\end{aligned}\tag{3.46}$$

For the second term, we need to use Lemma 3.2 in addition to the other lemmas:

$$\begin{aligned}
&\rho_{i,j}|\mathcal{H}(\theta_\mathcal{E}\delta^\dagger_{i,s}(\mathcal{E})(w_{i,s}-\bar{w}_{i,s})|^2_{H^1(\Omega_{i,j})}\\
\le\ &\rho_{i,s}\delta^\dagger_{i,s}(\mathcal{E})^2|E^h_{is,ij}(\mathcal{H}(\theta_\mathcal{E}(w_{i,s}-\bar{w}_{i,s}))|^2_{H^1(\Omega_{i,j})}\\
\le\ &C\rho_{i,s}\delta^\dagger_{i,s}(\mathcal{E})^2||\mathcal{H}(\theta_\mathcal{E}(w_{i,s}-\bar{w}_{i,s})||^2_{H^1(\Omega_{i,s})}\\
\le\ &C\rho_{i,s}(1+log(H/h))^2|\mathcal{H}(w_{i,s})|^2_{H^1(\Omega_{i,s})}.
\end{aligned}\tag{3.47}$$

For the last term,

$$|\theta_\mathcal{E}(\bar{w}_{i,j}-\bar{w}_{i,s})|^2_{H^1(\Omega_{i,j})} = |\theta_\mathcal{E}|^2_{H^1(\Omega_{i,j})}|(\bar{w}_{i,j}-\bar{w}_{i,s})|^2.$$

The energy of $\theta_\mathcal{E}$ can be estimated using Lemma 3.5. Adding and subtracting the common value $w_{i,j}(\mathcal{V}) = w_{i,s}(\mathcal{V})$, where $\mathcal{V}$ is an end point of the edge $\mathcal{E}$, we find that

$$|(\bar{w}_{i,j}-\bar{w}_{i,s})|^2 \le 2|(\bar{w}_{i,j}-w_{i,j}(\mathcal{V}))|^2 + 2|(\bar{w}_{i,j}-w_{i,s}(\mathcal{V}))|^2.$$

We can estimate the first term on the right hand side using Lemma 3.3:

$$|(\bar{w}_{i,j}-w_{i,j}(\mathcal{V}))|^2 \le ||w_{i,j}-\bar{w}_{i,j}||^2_{L^2(\Omega_{i,j})} + C(1+log(H_s/h))|w_{i,j}|^2_{H^1(\Omega_{i,j})}.$$

The second term can be estimated in the same manner.

Vertex Terms

We note that

$$|\theta_{\mathcal{V}} v_{i,j}(\mathcal{V})|^2_{S^{(i,j)}} = \rho_{i,j}|\mathcal{H}(\theta_{\mathcal{V}} v_{i,j}(\mathcal{V}))|^2_{H^1(\Omega_{i,j})} = \rho_{i,j}|\theta_{\mathcal{V}} v_{i,j}(\mathcal{V})|^2_{H^1(\Omega_{i,j})}, \tag{3.48}$$

where the second equality follows from the fact that $v_{i,j}(\mathcal{V})$ is a constant and $\theta_{\mathcal{V}}$ is a discrete harmonic function. Denoting an auxiliary function which vanishes at every node in $\bar{\Omega}_{i,j,h}$ except at $\mathcal{V}$ where it assumes the value 1 by $\vartheta_{\mathcal{V}}$, we have

$$\begin{aligned} & \rho_{i,j}|\theta_{\mathcal{V}} v_{i,j}(\mathcal{V})|^2_{H^1(\Omega_{i,j})} \\ = \; & \rho_{i,j}|v_{i,j}(\mathcal{V})|^2|\theta_{\mathcal{V}}|^2_{H^1(\Omega_{i,j})} \\ \leq \; & \rho_{i,j}|v_{i,j}(\mathcal{V})|^2|\vartheta_{\mathcal{V}}|^2_{H^1(\Omega_{i,j})} \\ \leq \; & C\rho_{i,j}|v_{i,j}(\mathcal{V})|^2, \end{aligned} \tag{3.49}$$

where the first inequality follows from the minimality of the energy of the discrete harmonic functions and the second inequality from the fact that a nodal basis function in two dimensions has $O(1)$ energy. Using the formula (3.40) and Lemma 3.3,

$$\begin{aligned} & \rho_{i,j}|v_{i,j}(\mathcal{V})|^2 \\ = \; & \rho_{i,j}\left| w_i(\mathcal{V}) - \sum_{k\in\mathcal{N}_{\mathcal{V},gl}\setminus\{i\}} \delta_k^{\dagger}(\mathcal{V}) w_k(\mathcal{V}) \right|^2 \\ \leq \; & \rho_{i,j}|\mathcal{N}_{\mathcal{V},gl}| \left(|w_i(\mathcal{V})|^2 + \sum_{k\in\mathcal{N}_{\mathcal{V},gl}\setminus\{i\}} \delta_k^{\dagger}(\mathcal{V})^2 |w_k(\mathcal{V})|^2 \right) \\ \leq \; & \rho_{i,j}|\mathcal{N}_{\mathcal{V},gl}| \left(||\mathcal{H}(w_{i,j})||^2_{L^\infty(\Omega_{i,j})} + \sum_{k\in\mathcal{N}_{\mathcal{V},gl}\setminus\{i\}} \delta_k^{\dagger}(\mathcal{V})^2 ||\mathcal{H}(w_{k,l})||^2_{L^\infty(\Omega_{k,l})} \right) \\ \leq \; & C\rho_{i,j}|\mathcal{N}_{\mathcal{V},gl}|(1+log(H_s/h))^2 \Bigg(||\mathcal{H}(w_{i,j})||^2_{H^1(\Omega_{i,j})} + \\ & \sum_{k\in\mathcal{N}_{\mathcal{V},gl}\setminus\{i\}} \delta_k^{\dagger}(\mathcal{V})^2 ||\mathcal{H}(w_{k,l})||^2_{H^1(\Omega_{k,l})} \Bigg). \end{aligned} \tag{3.50}$$

And we proceed as before. □

We now present a condition number estimate for the FETI-FETI method. We denote the subspace of Lagrange multipliers in which the preconditioned conjugate gradient method is performed by V^k:

$$V^k := \{\lambda \in range(Z_\Gamma^k \widetilde{B}_{\Gamma_c} : Z_\Gamma^k \widetilde{B}_{\Gamma_c}\lambda \in range(\widetilde{S}_c)\}.$$

Recall that $M_D^{-1} := Z_\Gamma^k \widetilde{B}_{D,\Gamma_c}\widetilde{S}_c \widetilde{B}_{D,\Gamma_c}^T Z_\Gamma^k$. Also, let $F := Z_\Gamma^k \widetilde{B}_{\Gamma_c}\widetilde{S}_c^{\dagger}\widetilde{B}_{\Gamma_c}^T Z_\Gamma^k$. We then have the following result:

Theorem 3.10. *For any $\lambda \in V^k$, we have*

$$\langle M_D\lambda, \lambda\rangle \leq \langle F\lambda, \lambda\rangle \leq C\frac{H_b}{H_s}(1+log(H_s/h))^2\langle M_D\lambda, \lambda\rangle,$$

where $C > 0$ is a constant independent of H_b, H_s, h.

Proof. With Lemma 3.9, the proof proceeds exactly the same as the proof of [30, Theorem 6.15]. □

In Section 4, we will present numerical results which show that the the algebraic factor H_b/H_s in Theorem 3.10 cannot be removed.

Table 1: Results for the FETI-FETI method. cond and iter denote condition number estimates and the iteration counts, respectively. Columns (I) and (II) indicate the areas on which continuity is imposed between the bodies. In Column (I), Γ, i.e., the entire interface, is considered. In Column (II), Γ_0, a proper subset of Γ, is considered.

			(I)		(II)	
$1/H_b$	H_b/H_s	H_s/h	cond	iter	cond	iter
2	2	2	2.5536	7	1.9940	7
4			3.6188	12	2.8136	10
6			3.9929	13	2.8718	10
8			3.9004	13	2.7254	10
10			3.7063	13	2.6951	10
12			4.0142	13	2.7227	10
2	4	2	7.1076	10	4.9790	9
	6		12.07490	12	7.1625	10
	8		17.1343	13	7.7988	10
	10		22.2380	15	8.6543	11
	12		27.3672	14	12.2114	12
	14		32.5125	17	12.0330	12
	16		37.6688	19	15.4197	12
	18		42.8328	20	16.4836	12
2	2	4	4.5201	9	4.2654	9
		8	6.9059	10	6.3238	10
		16	9.8320	12	9.1107	12
		32	13.2897	13	12.1263	13
		64	17.2765	16	15.6644	14
		128	21.7917	18	23.3692	17

4 Numerical Experiments

Recall that an active set method consists of outer iterations, in which the active set is updated, and inner iterations, in which auxiliary equality constrained problems are solved on the current active set. In this section, we solve such auxiliary equality constrained problems using the FETI-FETI method.

We solve the following minimization problem:

$$\min \sum_{i=1}^{N_b \times N_b} \left(\frac{1}{2} \int_{\Omega^i} |\nabla u^i|^2 dx - \int_{\Omega^i} f u^i dx \right), \tag{4.1}$$

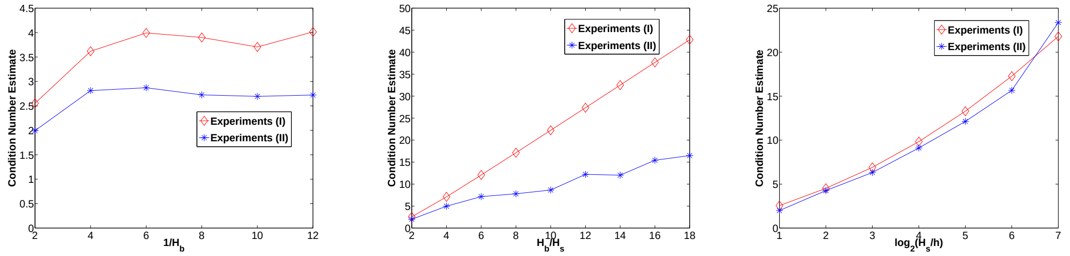
where $\Omega_i \subset \mathbb{R}^2, i = 1, \cdots, N_b \times N_b$ are square bodies with side length $H_b := 1/N_b$ which form the system $\Omega = \bigcup_{i=1}^{N_b \times N_b} \Omega_i = [0,1] \times [0,1]$. We require $u^i \in H^1(\Omega_i), u^i|_{\partial\Omega_i \cap \partial\Omega} = 0$. Each Ω_i is decomposed into $N_s \times N_s$ square subdomains, each of which is discretized by square bilinear elements of side length h. Also, $\Gamma := \cup_{i \neq j} \partial\Omega_i \cap \partial\Omega_j$ denotes the interface between the bodies.

We consider two linearized problems, with a different *contact area* between the bodies. In the first problem, the entire Γ is considered as the contact area, i.e., we require the continuity of the displacement vector across the entire Γ. In the second problem, continuity is imposed only on the middle third of the faces between the bodies. The Krylov subspace method of choice is the preconditioned conjugate gradient method for the FETI-FETI method. All our experiments have been performed in MATLAB, and the stopping criterion is $||r_n||_2/||r_0||_2 < 10^{-6}$, where r_n and r_0 are the n th and initial residuals, respectively.

In Table 1, the results obtained with the FETI-FETI method are presented. We have three parameters: the number of bodies across Ω ($N_b = 1/H_b$), the number of subdomains across each body ($N_s = H_b/H_s$), and the number of elements across each subdomain (H_s/h). We vary one parameter while keeping the other two fixed. The results for the first set of experiments, with the entire Γ as the contact surface, are shown in Column (I); those for the second set of experiments with a reduced contact area shown in Column (II). We observe that the condition number estimates and the iteration count are independent of $1/H_b$, depend linearly on H_b/H_s, and depend logarithmically on H_s/h. The condition numbers from Table 1 are also plotted in Figure 3.

Similar results have been obtained independently by Klawonn and Rheinbach; see [19] and [20].

Figure 3: Condition number estimates for the FETI-FETI method. Area on which continuity is imposed between bodies: Γ, i.e., the entire interface for (I), and only a proper subset of Γ, Γ_0 for (II)



(a) $1/H_b$ varies

(b) H_b/H_s varies

(c) H_s/h varies

References

[1] P. Avery, G. Rebel, M. Lesoinne, and C. Farhat, *A numerically scalable dual-primal substructuring method for the solution of contact problems–part I: the frictionless case*, Comput. Methods Appl. Mech. Engrg., 193 (2004), pp. 2403–2426.

[2] M. Bhardwaj, D. Day, C. Farhat, M. Lesoinne, K. Pierson, and D. Rixen, *Application of the FETI method to ASCI problems - scalability results on one thousand processors and discussion of highly heterogeneous problems*, Internat. J. Numer. Methods Engrg., 47 (2000), pp. 513–535.

[3] J. H. Bramble, J. E. Pasciak, and A. H. Schatz, *The construction of preconditioners for elliptic problems by substructuring. I*, Math. Comp., 47 (1986), pp. 103–134.

[4] S. C. Brenner and L. R. Scott, *The mathematical theory of finite element methods*, vol. 15 of Texts in Applied Mathematics, Springer, New York, third ed., 2008.

[5] S. C. Brenner and L.-Y. Sung, *Discrete Sobolev and Poincaré inequalities via Fourier series*, East-West J. Numer. Math., 8 (2000), pp. 83–92.

[6] C. R. Dohrmann, A. Klawonn, and O. B. Widlund, *Domain decomposition for less regular subdomains: overlapping Schwarz in two dimensions*, SIAM J. Numer. Anal., 46 (2008), pp. 2153–2168.

[7] Z. Dostál, *Optimal quadratic programming algorithms. With applications to variational inequalities.*, vol. 23 of Springer Optimization and Its Applications, Springer, New York, 2009.

[8] M. Dryja and O. B. Widlund, *Some domain decomposition algorithms for elliptic problems*, in Iterative methods for large linear systems (Austin, TX, 1988), Academic Press, Boston, MA, 1990, pp. 273–291.

[9] R. G. Durán and M. A. Muschietti, *The Korn inequality for Jones domains*, Electron. J. Differential Equations, (2004), pp. 1–10.

[10] C. Farhat, P.-S. Chen, and J. Mandel, *A scalable lagrange multiplier based domain decomposition method for time-dependent problems*, Internat. J. Numer. Methods Engrg., 38 (1995), pp. 3831–3853.

[11] C. Farhat, P.-S. Chen, F. Risler, and F.-X. Roux, *A unified framework for accelerating the convergence of iterative substructuring methods with Lagrange multipliers*, Internat. J. Numer. Methods Engrg., 42 (1998), pp. 257–288.

[12] C. Farhat and M. Géradin, *On the general solution by a direct method of a large-scale singular system of linear equations: application to the analysis of floating structures*, Internat. J. Numer. Methods Engrg., 41 (1998), pp. 675–696.

[13] C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, and D. Rixen, *FETI-DP: a dual-primal unified FETI method. I. A faster alternative to the two-level FETI method*, Internat. J. Numer. Methods Engrg., 50 (2001), pp. 1523–1544.

[14] C. Farhat, J. Mandel, and F.-X. Roux, *Optimal convergence properties of the FETI domain decomposition method*, Comput. Methods Appl. Mech. Engrg., 115 (1994), pp. 365–385.

[15] C. Farhat and F.-X. Roux, *A method of finite element tearing and interconnecting and its parallel solution algorithm*, Internat. J. Numer. Methods Engrg., 32 (1991), pp. 1205–1227.

[16] C. Farhat and F.-X. Roux, *Implicit parallel processing in structural mechanics*, in Computational Mechanics Advances, J. T. Oden, ed., vol. 2 (1), North-Holland, 1994, pp. 1–124.

[17] A. Klawonn and O. Rheinbach, *A parallel implementation of dual-primal FETI methods for three-dimensional linear elasticity using a transformation of basis*, SIAM J. Sci. Comput., 28 (2006), pp. 1886–1906.

[18] ——, *Robust FETI-DP methods for heterogeneous three dimensional elasticity problems*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 1400–1414.

[19] ——, *A hybrid approach to 3-level FETI*, PAMM Proc. Appl. Math. Mech., 8 (2008), pp. 10841–10843.

[20] ——, *Highly scalable parallel domain decomposition methods with an application to biomechanics*, ZAMM Z. Angew. Math. Mech., 90 (2010), pp. 5–32.

[21] A. Klawonn, O. Rheinbach, and O. B. Widlund, *An analysis of a FETI-DP algorithm on irregular subdomains in the plane*, SIAM J. Numer. Anal., 46 (2008), pp. 2484–2504.

[22] A. Klawonn and O. B. Widlund, *FETI and Neumann-Neumann iterative substructuring methods: connections and new results*, Comm. Pure Appl. Math., 54 (2001), pp. 57–90.

[23] ——, *Dual-primal FETI methods for linear elasticity*, Comm. Pure Appl. Math., 59 (2006), pp. 1523–1572.

[24] A. Klawonn, O. B. Widlund, and M. Dryja, *Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients*, SIAM J. Numer. Anal., 40 (2002), pp. 159–179.

[25] J. Lee, *A Hybrid Domain Decomposition Method and its Applications to Contact Problems*, PhD thesis, Courant Institute of Mathematical Sciences, September 2009.

[26] J. Li and O. B. Widlund, *FETI-DP, BDDC, and block Cholesky methods*, Internat. J. Numer. Methods Engrg., 66 (2006), pp. 250–271.

[27] J. Mandel and R. Tezaur, *Convergence of a substructuring method with Lagrange multipliers*, Numer. Math., 73 (1996), pp. 473–487.

[28] ——, *On the convergence of a dual-primal substructuring method*, Numer. Math., 88 (2001), pp. 543–558.

[29] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Éditeurs, Paris, 1967.

[30] A. Toselli and O. Widlund, *Domain decomposition methods—algorithms and theory*, vol. 34 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 2005.

[31] P. Wriggers, *Computational Contact Mechanics*, Wiley, New York, 2002.