Solution of the PDE Midterm

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**Problem 1.** In this problem you will study space-time rescaling of the viscous Burgers’ equation

\[ u_t + uu_x = u_{xx}. \]

Show that if \( u(x, t) \) is a solution, \( u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t) \) is also a solution, where \( \lambda \) is an arbitrary parameter.

**Solution.** By chain rule,

\[
\begin{align*}
\partial_t u_\lambda(x, t) &= \lambda^3 \partial_t u(\lambda x, \lambda^2 t), \\
\partial_x u_\lambda(x, t) &= \lambda^2 \partial_x u(\lambda x, \lambda^2 t), \\
\partial_{xx} u_\lambda(x, t) &= \lambda^3 \partial_{xx} u(\lambda x, \lambda^2 t).
\end{align*}
\]

Hence,

\[
\partial_t u_\lambda(x, t) + u_\lambda(x, t) \partial_x u_\lambda(x, t) = \lambda^3 \partial_t u(\lambda x, \lambda^2 t) + \lambda u(\lambda x, \lambda^2 t) \cdot \lambda^2 \partial_x u(\lambda x, \lambda^2 t)
\]

\[= \lambda^3 [\partial_t u(\lambda x, \lambda^2 t) + u(\lambda x, \lambda^2 t) \partial_x u(\lambda x, \lambda^2 t)]
\]

\[= \lambda^3 \partial_{xx} u(\lambda x, \lambda^2 t)
\]

\[= \partial_{xx} u_\lambda(x, t).
\]

This proves that \( u_\lambda \) is also a solution for viscous Burgers’ equation for \( \forall \lambda \in \mathbb{R} \).

**Remark.** A typical mistake in this problem is missing the blue \( \lambda \) above, given by the factor \( u_\lambda \).

\[ \square \]

**Problem 2.**

(a) Solve the PDE

\[ u_t + (1 + t^2)u_x = 0 \]

on the whole real line, with the initial condition \( u(x, 0) = \phi(x) \).

(b) Sketch the characteristic curves.

(c) Now add a source term on the right hand side,

\[ u_t + (1 + t^2)u_x = f(x, t) \]

and solve the equation again with the same initial condition \( u(x, 0) = \phi(x) \).

**Solution.**

(a) Assume that a characteristic curve is given by \( \gamma(s) = (x(s), t(s)) \). Let \( u(s) = u(x(s), t(s)) \). By chain rule, we have

\[
\frac{du}{ds} = u_t \frac{dt}{ds} + u_x \frac{dx}{ds}.
\]

It is natural to take

\[
\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = 1 + t^2,
\]

(1)
which gives
\[ t(s) = s + C_1. \]
Without loss of generality, we may assume \( C_1 = 0 \), to get \( t = s \) and then from \( dx/dt = 1 + t^2 \) we get that the equation for the characteristics is
\[ x = t + \frac{t^3}{3} + x_0, \]
where \( x_0 \) is the starting point of the characteristics at \( t = 0 \). With \( x(s) \) and \( t(s) \) satisfying (1), we know that \( \frac{du}{ds} = 0 \), i.e., \( u(s) \) is constant along the characteristics. Hence,
\[ u(x(s), t(s)) = u(x(0), t(0)) = u(x(0), 0) = \phi(x_0) = \phi\left(x - t - \frac{t^3}{3}\right). \]
The solution is therefore
\[ u(x, t) = \phi\left(x - t - \frac{t^3}{3}\right). \]

(b) The equations for characteristics are
\[ x(t) = t + \frac{t^3}{3} + x_0. \]
The graph is omitted here.

\textit{Remark.} You can put either \( t \)-axis or \( x \)-axis to be horizontal. However, the curves need to match the coordinate you choose. It would be a typical mistake, for example, if one puts \( x \)-axis to be horizontal but draws the curve to be like the typical graph of a cubic function. You can extend the characteristics into negative-\( t \) region — no problem but not really necessary here. However, it is suggested to sketch the characteristics into negative-\( x \) region. This is because the equation is defined for \( x \in \mathbb{R} \) instead of \( x \in \mathbb{R}_+ \). It is not an initial-boundary value problem, and there is no reason to sketch characteristics only for \( x > 0 \).

(c) Let \( F_t \) be the forward map from 0 to \( t \) defined by
\[ F_t(\psi)(x) = v(x, t), \]
where \( v(x, t) \) solves the homogeneous problem in (a) with non-zero initial data \( \psi \). By (a), we know that
\[ F_t(\psi)(x) = \psi\left(x - t - \frac{t^3}{3}\right). \]
Hence, for any \( 0 < s < t \), the forward map from \( s \) to \( t \) is given by
\[ F_{s,t}(\psi)(x) = F_t \circ F_{t-s}^{-1}(\psi)(x) = \psi\left(x - t - \frac{t^3}{3} + s + \frac{s^3}{3}\right). \]

Hence, by Duhamel’s formula, the solution for the inhomogenous problem writes
\[
u(x, t) = F_t(\phi) + \int_0^t F_{s,t}f(x, s) \, ds = \phi\left(x - t - \frac{t^3}{3}\right) + \int_0^t f\left(x - t - \frac{t^3}{3} + s + \frac{s^3}{3}, s\right) \, ds. \]

\textit{Remark.} A typical mistake here is to mistakenly use
\[ F_{s,t}(\psi)(x) = F_{t-s}(\psi)(x). \]
This is not true for equations that are not translation invariant in \( t \)!
Another approach is using the methods of characteristics. Having the same characteristics as in (a), i.e.,
\[ t(s) = s, \quad x(s) = s + \frac{s^3}{3} + x_0, \]
we know that
\[ \frac{du}{ds}(s) = f(x(s), t(s)). \]
Solving this ODE gives
\[ u(s) = u(0) + \int_0^s f(x(\tau), t(\tau)) \, d\tau. \] (2)

Therefore the solution becomes
\[ u(x, t) = u(x_0, 0) + \int_0^t f(x_0 + \tau + \frac{\tau^3}{3}, t) \, d\tau \]
\[ = \phi \left( x - t - \frac{t^3}{3} \right) + \int_0^t f \left( x - t - \frac{t^3}{3} + \tau + \frac{\tau^3}{3}, t \right) \, d\tau. \]

**Remark.** Mistakes could be made, for example, if one writes the following equation in the place of (2)
\[ u(x, t) = u(x_0, 0) + \int_0^t f(x, \tau) \, d\tau. \]
Note that we used \( t = \tau \) here. In this way, the first argument of \( f \) in the integral seems to be independent of \( \tau \), which is not true. It is always suggested to write a more explicit form like (2), without omitting any dependence on the parameterization along the characteristics. Another good way to avoid this type of mistake is to keep in mind that one always solves ODEs *along* the characteristics. Therefore, the integral of \( f \) should be done along the characteristics as well. Recall that the equation of the characteristics going through \( (x_*, t_*) \) is that
\[ x - t - \frac{t^3}{3} = x_0 = x_* - t_* - \frac{t_*^3}{3}, \]
or in a parameterized form
\[ (x(s), t(s)) = \left( s + \frac{s^3}{3} + x_0, s \right) = \left( s + \frac{s^3}{3} + x_* - t_* - \frac{t_*^3}{3}, s \right). \]
In this way, one can tell why the following representation of the final solution is not correct
\[ u(x, t) = \phi \left( x - t - \frac{t^3}{3} \right) + \int_0^t f \left( x - t - \frac{t^3}{3} + \tau + \frac{\tau^3}{3}, \tau \right) \, d\tau. \]
Note that \( (x - t - \frac{t^3}{3}, \tau) \) is not on the characteristics that goes through \( (x, t) \).

**Problem 3.** Consider the PDE
\[ u_{xx} - 3u_{xt} - 4u_{tt} = 0 \] (3)
on the whole real line. Classify this equation (e.g., first/second/third order linear/nonlinear elliptic/parabolic/hyperbolic equation), and then solve it with the initial conditions
\[ u(x, 0) = 2x^2, \quad u_t(x, 0) = x. \]
Solution. The equation is a second-order linear equation with
\[ A = 1, \quad B = -3, \quad C = -4. \]
The discriminant
\[ D = B^2 - 4AC = 9 + 16 = 25 > 0, \]
which implies that (3) is a hyperbolic equation. By factorizing the differential operator, we can rewrite the equation as
\[ (\partial_x + \partial_t)(\partial_x - 4\partial_t)u = 0. \tag{4} \]
To solve (3), we look for change of variables
\[ x = a\eta + b\xi, \quad t = c\eta + d\xi. \]
We hope that under the new \((\eta, \xi)-coordinate, (4) becomes
\[ \partial_{\eta\xi}u = 0. \tag{5} \]
To achieve this, we compute by chain rule,
\[
\begin{align*}
\frac{\partial}{\partial \eta} &= \frac{\partial x}{\partial \eta} \frac{\partial}{\partial x} + \frac{\partial t}{\partial \eta} \frac{\partial}{\partial t} = a \frac{\partial}{\partial x} + c \frac{\partial}{\partial t}, \\
\frac{\partial}{\partial \xi} &= \frac{\partial x}{\partial \xi} \frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi} \frac{\partial}{\partial t} = b \frac{\partial}{\partial x} + d \frac{\partial}{\partial t}.
\end{align*}
\]
Hence,
\[ \partial_{\eta\xi}u = (a\partial_x + c\partial_t)(b\partial_x + d\partial_t)u. \tag{6} \]
Comparing (6) with (4), it is natural to choose
\[ a = 1, \quad c = 1, \quad b = 1, \quad d = -4, \]
i.e.,
\[ x = \eta + \xi, \quad t = \eta - 4\xi, \]
or equivalently,
\[ \eta = \frac{4x + t}{5}, \quad \xi = \frac{x - t}{5}. \tag{7} \]
Remark. A trick to quickly find out the correct characteristic coordinate for (4) is to use the fact that the characteristic coordinate corresponding to the first order differential operater \(\partial_t + c\partial_x\) is given by \(x - ct\). In our case, this gives \(x - t\) and \(x + \frac{1}{4}t\), which is equivalent to (7).
However, if one applies the chain rule carelessly or goes too fast through the above trick, mistakes could be made — coefficients in the change of variables may have wrong signs or somehow get swapped.

Now under this change of variable the equation becomes (5). We simply integrate (5) in \(\eta\) and then \(\xi\) to find
\[ \partial_{\eta\xi}u = 0 \Rightarrow \partial_\xi u = f(5\xi), \]
\[ \Rightarrow u = F(5\xi) + G(5\eta). \]
with \(F\) and \(G\) to be determined. Note that we put a factor of 5 in the argument, simply because this can give a simpler form of general solution of \(u\) below. Then by (7), in \((x, t)\)-coordinate, this becomes
\[ u(x, t) = F(x - t) + G(4x + t). \tag{8} \]
Take $x$- and $t$-derivatives and we find
\[
\begin{align*}
u_x(x,t) &= F'(x-t) + 4G'(4x+t), \\
u_t(x,t) &= -F'(x-t) + G'(4x+t).
\end{align*}
\]

Putting $t = 0$ and using the initial condition, we find
\[
\begin{align*}
4x &= F'(x) + 4G'(4x), \\
x &= -F'(x) + G'(4x).
\end{align*}
\]
This gives
\[
F'(x) = 0, \quad G'(4x) = x. \tag{9}
\]
The second equation above could be rewritten as
\[
G'(x) = \frac{x}{4}.
\]
By solving the equation for $F$ and $G$, we find
\[
F(x) = C_1, \quad G(x) = \frac{x^2}{8} + C_2.
\]

**Remark.** A typical mistakes in finding out $F$ and $G$ is trying to integrate $G'(4x) = x$ in $t$. Since $F$ and $G$ are functions on $\mathbb{R}$, $F'$ and $G'$ really mean their derivatives with respect to their own arguments (no matter what they are). In this case, they are just $x$ or multiples of $x$. Also, mistakes could be made if one tries to integrate $G'(4x) = x$ but forgets to apply the chain rule to deal with the factor 4.

Therefore, using (9),
\[
u(x,t) = \frac{(4x + t)^2}{8} + C.
\]
Let $t = 0$ and we will find $C = 0$. Hence, the solution is
\[
u(x,t) = \frac{(4x + t)^2}{8}.
\]

**Problem 4.** Let $G(x,t)$ denote the Green’s function for the heat equation $u_t = ku_{xx}$ on the whole real line (one dimension).

(a) Show that $G_2(x,y,t) = G(x,t)G(y,t)$ solves the heat equation in two dimensions $u_t = k(u_{xx} + u_{yy})$ on the whole $x - y$ plane.

(b) Convert $G_2(x,y,t)$ to polar coordinates and show that it is only a function of the distance $r = \sqrt{x^2 + y^2}$, i.e., that $G_2(x,y,t) \equiv G_r(r,t)$

(c) What initial condition does the solution $G_2(x,y,t)$ satisfy? Explain why $G_2(x,y,t)$ is the Green’s function for the heat equation in two dimensions.

(d) (Extra credit points) Generalize parts (a), (b) and (c) above to an arbitrary number of dimensions $d$.

**Remark.** In this problem, $u$ is not a function we need to take care. The equation $u_t = k(u_{xx} + u_{yy})$ above just tells you what equation we are discussing, where $u$ acts like a “dummy function”. It is not an initial-value problem of the heat equation one needs to solve.
Solution. (a) By definition,
\[
\partial_t G_2(x, y, t) = \partial_t G(x, t)G(y, t) + G(x, t)\partial_t G(y, t)
\]
\[
= k\partial_{xx} G(x, t)G(y, t) + G(x, t) \cdot k\partial_{yy} G(y, t)
\]
\[
= k\partial_{xx}(G(x, t)G(y, t)) + k\partial_{yy}(G(x, t)G(y, t))
\]
\[
= k(\partial_{xx} + \partial_{yy})G_2(x, y, t)
\]

Remark. Of course, one can plug in the formula for \(G(x, t)\) and \(G(y, t)\) and take partial derivatives to check, but that makes an easy problem painful. Always keep in mind that \(G(x, t)\), as Green’s function, satisfies 1D heat equation.

(b) Recall that the Green’s function for the 1D heat equation \(u_t = ku_{xx}\) is that
\[
G(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}.
\]
Hence,
\[
G_2(x, y, t) = G(x, t)G(y, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \cdot \frac{1}{\sqrt{4\pi kt}} e^{-\frac{y^2}{4kt}} = \frac{1}{4\pi kt} e^{-\frac{x^2+y^2}{4kt}} = \frac{1}{4\pi kt} e^{-\frac{r^2}{4kt}},
\]
which is a function of \(r \) and \(t \).

(c) Recall that
\[
\delta(x) = \lim_{t \to 0^+} G(x, t).
\]
The convergence is understood in distribution, i.e., for any test function \(\phi(x) \in C_0^\infty(\mathbb{R})\),
\[
\int_\mathbb{R} \delta(x)\phi(x) \, dx = \phi(0) = \lim_{t \to 0^+} \int_\mathbb{R} G(x, t)\phi(x) \, dx.
\]
By definition of \(G_2(x, y, t)\), for \(\forall \phi(x, y) \in C_0^\infty(\mathbb{R}^2)\), it is not difficult to check
\[
\lim_{t \to 0^+} \int_{\mathbb{R}^2} G_2(x, y, t)\phi(x, y) \, dxdy = \phi(0, 0).
\]
And we know
\[
\phi(0, 0) = \int_{\mathbb{R}^2} \delta(x)\delta(y)\phi(x, y) \, dxdy.
\]
Therefore,
\[
\lim_{t \to 0^+} G_2(x, y, t) = \delta(x)\delta(y).
\]
Since it has been shown in part (a) that \(G_2\) satisfies 2D heat equation \(\partial_t G_2 = k(\partial_{xx} + \partial_{yy})G_2\), \(G_2(x, y, t)\) is the Green’s function of the 2D heat equation.

(d) In \(d\)-dimension, let \(x = (x_1, \ldots, x_d) \in \mathbb{R}^d\). We define
\[
G_d(x, t) = \Pi_{i=1}^d G(x_i, t).
\]
In a similar manner, with details omitted here, we are able to show \(G_d\) satisfies the heat equation \(\partial_t G_d = k\Delta G_d\) in \(d\)-dimension.
\[
G_d(x, t) = \frac{1}{(4\pi kt)^{d/2}} e^{-\frac{|x|^2}{4kt}},
\]
which is a radial function. And
\[
G_d(x, t) \to \delta^d(x) \triangleq \Pi_{i=1}^d \delta(x_i) \quad \text{as} \ t \to 0^+.
\]
Hence, \(G_d(x, t)\) is the Green’s function for heat equation in \(d\)-dimension and it is radial.