Solution of the PDE midterm

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Problem 1. In this problem you will study space-time rescaling of the viscous Burgers’ equation

\[ u_t + uu_x = u_{xx}. \]

Show that if \( u(x, t) \) is a solution, \( u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t) \) is also a solution, where \( \lambda \) is an arbitrary parameter.

Solution. By chain rule,

\[
\begin{align*}
\partial_t u_\lambda(x, t) &= \lambda^3 \partial_t u(\lambda x, \lambda^2 t), \\
\partial_x u_\lambda(x, t) &= \lambda^2 \partial_x u(\lambda x, \lambda^2 t), \\
\partial_{xx} u_\lambda(x, t) &= \lambda^3 \partial_{xx} u(\lambda x, \lambda^2 t).
\end{align*}
\]

Hence,

\[
\begin{align*}
\partial_t u_\lambda(x, t) + u_\lambda(x, t) \partial_x u_\lambda(x, t) &= \lambda^3 \partial_t u(\lambda x, \lambda^2 t) + \lambda u(\lambda x, \lambda^2 t) \cdot \lambda^2 \partial_x u(\lambda x, \lambda^2 t) \\
&= \lambda^3 \partial_{xx} u(\lambda x, \lambda^2 t) \\
&= \partial_{xx} u_\lambda(x, t).
\end{align*}
\]

This proves that \( u_\lambda \) is also a solution for viscous Burgers’ equation for \( \forall \lambda \in \mathbb{R} \). \( \square \)

Problem 2. (a) Solve the PDE

\[ u_t + (1 + t^2)u_x = 0 \]

on the whole real line, with the initial condition \( u(x, 0) = \phi(x) \).

(b) Sketch the characteristic curves.

(c) Now add a source term on the right hand side,

\[ u_t + (1 + t^2)u_x = f(x, t) \]

and solve the equation again with the same initial condition \( u(x, 0) = \phi(x) \).

Solution. (a) Assume that a characteristic curve is given by \( \gamma(s) = (x(s), t(s)) \). Let \( u(s) = u(x(s), t(s)) \). By chain rule, we have

\[
\frac{du}{ds} = \frac{du}{dt} \frac{dt}{ds} + \frac{du}{dx} \frac{dx}{ds}.
\]

It is natural to take

\[
\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = 1 + t^2, 
\]

which gives

\[ t(s) = s + C_1. \]
Without loss of generality, we may assume $C_1 = 0$, to get $t = s$ and then from $dx/dt = 1 + t^2$ we get that the equation for the characteristics is
\[
x = t + \frac{t^3}{3} + x_0,
\]
where $x_0$ is the starting point of the characteristics at $t = 0$. With $x(s)$ and $t(s)$ satisfying (1), we know that $\frac{du}{ds} = 0$, i.e., $u(s)$ is constant along the characteristics. Hence,
\[
u(x(s), t(s)) = u(x(0), t(0)) = u(x(0), 0) = \phi(x_0) = \phi\left(x - t - \frac{t^3}{3}\right).
\]
The solution is therefore
\[
u(x, t) = \phi\left(x - t - \frac{t^3}{3}\right).
\]
(b) Omitted.
(c) Let $\mathcal{F}_t$ be the forward map from 0 to $t$ defined by
\[
\mathcal{F}_t(\psi)(x) = v(x, t),
\]
where $v(x, t)$ solves the homogeneous problem in (a) with non-zero initial data $\psi$. By (a), we know that
\[
\mathcal{F}_t(\psi)(x) = \psi\left(x - t - \frac{t^3}{3}\right).
\]
Hence, for any $0 < s < t$, the forward map from $s$ to $t$ is given by
\[
\mathcal{F}_{s,t}(\psi)(x) = \mathcal{F}_t \circ \mathcal{F}_s^{-1}(\psi)(x) = \psi\left(x - t - \frac{t^3}{3} + s + \frac{s^3}{3}\right).
\]
Hence, by Duhamel’s formula, the solution for the inhomogenous problem writes
\[
u(x, t) = \mathcal{F}_t(\phi) + \int_0^t \mathcal{F}_{s,t} f(x, s) \, ds
= \phi\left(x - t - \frac{t^3}{3}\right) + \int_0^t f\left(x - t - \frac{t^3}{3} + s + \frac{s^3}{3}, s\right) \, ds.
\]
Another approach is naively using the methods of characteristics. Using the same characteristics as in (a), i.e.,
\[
t(s) = s, \quad x(s) = s + \frac{s^3}{3} + x_0,
\]
we know that
\[
\frac{du}{ds}(s) = f(x(s), t(s)).
\]
Solving this ODE gives
\[
u(s) = u(0) + \int_0^s f(x(\tau), t(\tau)) \, d\tau.
\]
Therefore the solution becomes
\[
u(x, t) = u(x_0, 0) + \int_0^t f\left(x_0 + \tau + \frac{\tau^3}{3}, \tau\right) \, d\tau
= \phi\left(x - t - \frac{t^3}{3}\right) + \int_0^t f\left(x - t - \frac{t^3}{3} + \tau + \frac{\tau^3}{3}, \tau\right) \, d\tau.
\]
Problem 3. Consider the PDE
\[ u_{xx} - 3u_{xt} - 4u_{tt} = 0 \]  
(2)
on the whole real line. Classify this equation (e.g., first/second/third order linear/nonlinear elliptic/parabolic/hyperbolic equation), and then solve it with the initial conditions
\[ u(x, 0) = 2x^2, \quad u_t(x, 0) = x. \]

Solution. The equation is a second-order linear equation with
\[ A = 1, \quad B = -3, \quad C = -4. \]
The discriminant
\[ D = B^2 - 4AC = 9 + 16 = 25 > 0, \]
which implies that (2) is a hyperbolic equation. By factorizing the differential operator, we can rewrite the equation as
\[ (\partial_x + \partial_t)(\partial_x - 4\partial_t)u = 0. \]  
(3)
To solve (2), we look for change of variables
\[ x = a\eta + b\xi, \]
\[ t = c\eta + d\xi. \]
We hope that under the new \((\eta, \xi)\)-coordinate, (3) becomes
\[ \partial_{\eta\xi}u = 0. \]  
(4)
To achieve this, we compute by chain rule,
\[ \frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta} \frac{\partial}{\partial x} + \frac{\partial t}{\partial \eta} \frac{\partial}{\partial t} = a \frac{\partial}{\partial x} + c \frac{\partial}{\partial t}, \]
\[ \frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi} \frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi} \frac{\partial}{\partial t} = b \frac{\partial}{\partial x} + d \frac{\partial}{\partial t}. \]
Hence,
\[ \partial_{\eta\xi}u = (a\partial_x + c\partial_t)(b\partial_x + d\partial_t)u. \]  
(5)
Comparing (5) with (3), it is natural to choose
\[ a = 1, \quad c = 1, \quad b = 1, \quad d = -4, \]
i.e.,
\[ x = \eta + \xi, \quad t = \eta - 4\xi, \]
or equivalently,
\[ \eta = \frac{4x + t}{5}, \quad \xi = \frac{x - t}{5}. \]  
(6)
Now under this change of variable the equation becomes (4). We simply integrate (4) in \(\eta\) and then \(\xi\) to find
\[ \partial_{\eta\xi}u = 0 \Rightarrow \partial_\xi u = f(5\xi), \]
\[ \Rightarrow u = F(5\xi) + G(5\eta). \]
with \(F\) and \(G\) to be determined. Note that we put a factor of 5 in the argument, simply because this can give a simpler form of general solution of \(u\) below. Then by (6), in \((x, t)\)-coordinate, this becomes
\[ u(x, t) = F(x - t) + G(4x + t). \]  
(7)
Take $x$- and $t$-derivatives and we find
\[
  u_x(x, t) = F'(x-t) + 4G'(4x+t), \\
  u_t(x, t) = -F'(x-t) + G'(4x+t).
\]

Putting $t = 0$ and using the initial condition, we find
\[
  4x = F'(x) + 4G'(4x), \\
  x = -F'(x) + G'(4x).
\]

This gives
\[
  F'(x) = 0, \quad G'(4x) = x.
\]

The second equation above could be rewritten as
\[
  G'(x) = \frac{x}{4}.
\]

By solving the equation for $F$ and $G$, we find
\[
  F(x) = C_1, \quad G(x) = \frac{x^2}{8} + C_2.
\]

Therefore, using (7),
\[
  u(x, t) = \frac{(4x + t)^2}{8} + C.
\]

Let $t = 0$ and we will find $C = 0$. Hence, the solution is
\[
  u(x, t) = \frac{(4x + t)^2}{8}.
\]

\[\square\]

**Problem 4.** Let $G(x, t)$ denote the Green’s function for the heat equation $u_t = ku_{xx}$ on the whole real line (one dimension).

(a) Show that $G_2(x, y, t) = G(x, t)G(y, t)$ solves the heat equation in two dimensions $u_t = k(u_{xx} + u_{yy})$ on the whole $x-y$ plane.

(b) Convert $G_2(x, y, t)$ to polar coordinates and show that it is only a function of the distance $r = \sqrt{x^2 + y^2}$, i.e., that $G_2(x, y, t) \equiv G_r(r, t)$

(c) What initial condition does the solution $G_2(x, y, t)$ satisfy? Explain why $G_2(x, y, t)$ is the Green’s function for the heat equation in two dimensions.

(d) (Extra credit points) Generalize parts (a), (b) and (c) above to an arbitrary number of dimensions $d$.

**Solution.** (a) By definition,
\[
  \partial_t G_2(x, y, t) = \partial_t G(x, t)G(y, t) + G(x, t)\partial_t G(y, t)
  = k\partial_{xx}G(x, t)G(y, t) + G(x, t) \cdot k\partial_{yy}G(y, t)
  = k\partial_{xx}(G(x, t)G(y, t)) + k\partial_{yy}(G(x, t)G(y, t))
  = k(\partial_{xx} + \partial_{yy})G_2(x, y, t)
\]

(b) Recall that the Green’s function for the 1D heat equation $u_t = ku_{xx}$ is that
\[
  G(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4\pi kt}}.
\]

Hence,
\[
  G_2(x, y, t) = G(x, t)G(y, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4\pi kt}} \cdot \frac{1}{\sqrt{4\pi kt}} e^{-\frac{y^2}{4\pi kt}} = \frac{1}{4\pi kt} e^{-\frac{x^2+y^2}{4\pi kt}} = \frac{1}{4\pi kt} e^{-\frac{r^2}{4\pi kt}},
\]

which is a function of $r$ and $t$. 

(c) Recall that
\[ \delta(x) = \lim_{t \to 0^+} G(x, t). \]
The convergence is understood in distribution, i.e., for any test function \( \phi(x) \in C_0^\infty(\mathbb{R}) \),
\[ \int_{\mathbb{R}} \delta(x) \phi(x) \, dx = \phi(0) = \lim_{t \to 0^+} \int_{\mathbb{R}} G(x, t) \phi(x) \, dx. \]
By definition of \( G_2(x, y, t) \), for \( \forall \phi(x, y) \in C_0^\infty(\mathbb{R}^2) \), it is not difficult to check
\[ \lim_{t \to 0^+} \int_{\mathbb{R}^2} G_2(x, y, t) \phi(x, y) \, dxdy = \phi(0, 0). \]
And we know
\[ \phi(0, 0) = \int_{\mathbb{R}^2} \delta(x) \delta(y) \phi(x, y) \, dxdy. \]
Therefore,
\[ \lim_{t \to 0^+} G_2(x, y, t) = \delta(x) \delta(y). \]
Since it has been shown in part (a) that \( G_2 \) satisfies 2D heat equation \( \partial_t G_2 = k(\partial_{xx} + \partial_{yy}) G_2 \),
\( G_2(x, y, t) \) is the Green’s function of the 2D heat equation.

(d) In \( d \)-dimension, let \( x = (x_1, \cdots, x_d) \in \mathbb{R}^d \). We define
\[ G_d(x, t) = \Pi_{i=1}^d G(x_i, t). \]
In a similar manner, with details omitted here, we are able to show \( G_d \) satisfies the heat equation
\( \partial_t G_d = k \Delta G_d \) in \( d \)-dimension.
\[ G_d(x, t) = \frac{1}{(4\pi kt)^{d/2}} e^{-\frac{|x|^2}{4kt}}, \]
which is a radial function. And
\[ G_d(x, t) \to \delta^d(x) \triangleq \Pi_{i=1}^d \delta(x_i) \quad \text{as} \ t \to 0^+. \]
Hence, \( G_d(x, t) \) is the Green’s function for heat equation in \( d \)-dimension and it is radial. \( \square \)