Calculus - Day 3

October 25, 2007

Topics: Main topics: Lagrange multipliers, Implicit Function Theorem. Other topics: Implicit differentiation, normals/tangents to a surface/line, finding extrema, Taylor’s theorem, Taylor’s remainder, Mean Value Theorem, Fubini’s theorem.

Problems: J95#4, S03#4, S03#5, J95#3, S91#2, S94#4, S96#5, J01#1, J92#2.

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**Lagrange Multiplier** Suppose we would like to minimize/maximize $f(\vec{x})$ subject to constraints $g_1(\vec{x}) = 0$, $g_2(\vec{x}) = 0$, ... $g_m(\vec{x}) = 0$. If $x^*$ is a local minimizer/maximizer of the problem, and $\{\nabla g_i(x^*)\}$ are linearly independent, then $x^*$ is a critical point of the function

$$L(\vec{x}, \lambda_1, \lambda_2, ..., \lambda_m) = f(\vec{x}) - \lambda_1 g_1(\vec{x}) - ... - \lambda_m g_m(\vec{x}).$$

In other words, to find a minimizer we must solve the equations $\nabla_x L = 0$, $\nabla_\lambda L = 0$ for $x^*$, $\{\lambda_i\}$. We must also check points on the boundaries of the set, if such boundaries exist, as well as points where the constraints are degenerate.

**Implicit Function Theorem** Let $F = F(x, y, z) \in C^1$ on an open set $D$. Let $p_0 = (x_0, y_0, z_0) \in D$ satisfy $F(p_0) = 0$. Suppose $F_z(p_0) \neq 0$. Then there exists a function $\phi \in C^1$ defined in a neighbourhood $N$ of $(x_0, y_0)$ such that $z = \phi(x, y)$ solves $F(x, y, z) = 0$ for $(x, y) \in N$, and $\phi(x_0, y_0) = z_0$.

In general, if we have a system of $m$ equations in $n$ unknowns,

$$\Phi(x) = 0 \quad \Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m \in C^1,$$

and we know one solution $\Phi(p) = 0$, $p \in \mathbb{R}^n$, then we can solve for $m$ of the variables in terms of the others provided the Jacobian of the transformation doesn’t vanish at $p$:

$$(x_1, x_2, ..., x_m) = f(x_{m+1}, ..., x_n) \iff \frac{\partial(\phi_1, ..., \phi_m)}{\partial(x_1, ..., x_m)} \bigg|_p \neq 0.$$
Taylor’s Remainder  Let $f \in C^{n+1}$ on an interval $I$ about $x = c$, and let $P(x)$ be the Taylor polynomial of degree $n$ at $c$. Then $f(x) = P(x) + R_n(x)$ for $x \in I$, with

$$R_n(x) = \frac{1}{n!} \int_c^x f^{(n+1)}(t)(t-c)^n dt.$$  

Corollary:

$$R_n(x) = f^{(n+1)}(\tau)(x-c)^{n+1} \frac{1}{(n+1)!}$$

for some $\tau \in (c, x)$.

Fubini’s Theorem  If $\int_{A \times B} |f(x, y)| d(x, y) < \infty$, (integral with respect to product measure), then we can do the integration in any order: $\int_A \left( \int_B f(x, y) dy \right) dx = \int_B \left( \int_A f(x, y) dx \right) dy$. 

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