Advanced Calculus - Handout 1

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Topics: limits, series, sequences and series of functions, uniform convergence, power series, fundamental theorem of calculus, limits of integrals, improper integrals.

References: Buck Ch 4.5, 5, 6.1-6.4, selections from Ch 1-3. See Rudin (or other) for how to take limits of integrals.

Problem Set 1: S03#1, S94#4, J05#1, S93#2, S05#3(ii)

Problem Set 2: J07#2, S99#2, J06#1b, S98#3, S98#4, J98#4, J03#4

Extra Set 2: J06#1a

1 Limits

Useful techniques

• Hopital’s rule: if \( \lim f(x) = \lim g(x) = 0 \) or \( \lim f(x) = \lim g(x) = \infty \), then \( \lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} \)

• take logs: \( \lim f(x_n) = \exp(\lim \ln f(x_n)) \) (this works because \( a_n \to a \Rightarrow f(a_n) \to f(a) \) if \( f \) is continuous).

• use the \( e \)-limit: \( (1 + 1/n)^n \to e \) as \( n \to \infty \)

• multiply by the conjugate. eg: \( \sqrt{n^2 + n} - \sqrt{n^2 - n} = \frac{2n}{\sqrt{n^2} + \sqrt{n^2 - n}} \to 1 \)

2 Series

Types of convergence

• \( \sum a_n \) converges absolutely \( \Leftrightarrow \sum |a_n| < \infty \)

• \( \sum a_n \) converges conditionally \( \Leftrightarrow \sum a_n < \infty \) and \( \sum a_n \) does not converge absolutely
If $\sum a_n$ converges absolutely, the terms can be rearranged in any order and the rearranged series will converge to the same sum. If $\sum a_n$ converges conditionally, then the terms can be rearranged so that the resulting series converges to any desired number.

The following are some useful tests for convergence.

**Telescoping Series** If $a_n = A_{n+1} - A_n$ and $A_n \to A$, then $\sum a_n = A - A_1$.

**Comparison Test (+)** If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

A more sophisticated version of this is:

- $\limsup \frac{a_n}{b_n} < \infty$, $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges
- $\liminf \frac{a_n}{b_n} > 0$, $\sum b_n = \infty$ $\Rightarrow$ $\sum a_n = \infty$

**Ratio Test (+)** If $a_n \geq 0$, then

- $\limsup \frac{a_{n+1}}{a_n} < 1 \Rightarrow \sum a_n < \infty$
- $\liminf \frac{a_{n+1}}{a_n} > 0 \Rightarrow \sum a_n = \infty$

The test is inconclusive if $\lim \frac{a_{n+1}}{a_n} = 1$.

**Root Test (+)** If $a_n \geq 0$, then

- $\limsup a_n^{1/n} < 1 \Rightarrow \sum a_n < \infty$
- $\liminf a_n^{1/n} > 1 \Rightarrow \sum a_n = \infty$

The test is inconclusive if $\lim a_n^{1/n} = 1$.

**Integral Test (+)** If $a_n = f(n) \geq 0$ for $n \geq N$, and $f(x)$ is monotone decreasing on $[N, \infty)$, then $\sum a_n < \infty \Leftrightarrow \int_N^\infty f(x)dx < \infty$.

**Alternating Series Test (Leibniz Test)** If $|a_n|$ is monotonically decreasing and $\lim a_n = 0$, then $\sum (-1)^n |a_n| < \infty$.

**Dirichlet Test** If $a_n \to 0$, $\sum |a_{n+1} - a_n|$ converges, and $|\sum_{n=1}^N b_n| < M \forall N$, then $\sum a_n b_n$ converges.

A useful corollary: If $|\sum_{n=1}^N b_n| < M$, $a_n$ monotone and $a_n \downarrow 0$, then $\sum a_n b_n$ converges.

3 **Sequences and Series of Functions**

### 3.1 Uniform Convergence

(Throughout this section we will assume our functions are restricted to a domain $E \subseteq \mathbb{R}$, and convergence means convergence throughout this domain.)
**Definition** \( f_n(x) \to f(x) \) **uniformly** on \( E \subseteq \mathbb{R} \) means
\[
\|f_n - f\| \equiv \sup_{x \in E} |f_n(x) - f(x)| \to 0 \text{ as } n \to \infty.
\]
The following are some useful tests and theorems about uniform convergence.

**Weierstrass M-test** If \( \|f_n\| \leq M_n \) and \( \sum M_n < \infty \), then \( \sum f_n(x) \) converges uniformly. (Note: this test is a sufficient, but not a necessary test for convergence.)

**Dirichlet Test** If \( |\sum_{n=1}^N f_n(x)| < M, b_n \downarrow 0 \), then \( \sum f_n(x)b_n \) converges uniformly.

**Continuity, Integrability, Differentiability** of the limit
- (Continuity) If \( f_n \to f \) uniformly in \( E \) and \( f_n \) is continuous, then \( f \) is continuous.
- (Integrability) If \( f_n \to f \) uniformly in \( E = [a, b] \) and \( f_n \) is integrable, then \( f \) is integrable and \( \int f(x)dx = \lim_{n \to \infty} \int f_n(x)dx \).
- (Differentiability) If \( \sum f_n(x) \to S(x) \) uniformly in \( E = [a, b] \), \( f_n \) is differentiable with continuous derivatives and \( \sum f'_n(x) \) converges uniformly, then \( \sum f'_n(x) = S'(x) \).
- (Differentiability) If \( f_n(x) \) are differentiable with continuous derivatives, \( \sum f'_n(x) = \hat{S}'(x) \) uniformly, and \( \sum f_n(x_0) = S(x_0) \) at some point \( x_0 \in E \), then \( \sum f_n = S(x) \) uniformly in \( E \) and \( S'(x) = \hat{S}(x) \).

This has an extension to improper integrals - see Buck p.292, Thm 17.

### 3.2 Power Series
A series of the form \( \sum a_n x^n \) is called a **power series**.

Its **radius of convergence** is the positive constant \( R \) such that the power series converges absolutely for \( |x| < R \) and diverges for \( |x| > R \).

At \( |x| = R \) there may be convergence or divergence.

The series converges uniformly in \([−r, r]\) for \( r < R \).

If \( f(x) = \sum a_n x^n \) in \((-r, r), r > 0\), then \( a_n = \frac{f^{(n)}(0)}{n!} \).

Finding \( R \)
- \( R^{-1} = \lim \sup |a_n|^{1/n} \)
- \( R^{-1} = \lim \sup \frac{|a_{n+1}|}{|a_n|} \) (if this limit exists)

**Continuity, Integrability, and Differentiation**
- \( S(x) = \sum a_n x^n \) is continuous
- \( S'(x) = \sum na_n x^{n-1} \) uniformly in any compact interval contained in \((-R, R)\).
\[ \int_0^x S(t)\,dt = \sum \frac{a_n}{n+1} x^{n+1} \text{ uniformly in any cpt interval contained in } (-R, R). \]

Important Power Series

- \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \)
- \( \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \)
- \( \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \)
- \( \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \)

4 Integration Theorems

Theorems about interchanging the limit and the integral.

**Dominated Convergence Theorem** If \( f_n \to f \) pointwise, and there exists \( g(x) \geq 0 \) such that \( |f_n(x)|, |f(x)| \leq g(x) \) and \( \int g(x) < \infty \), then \( \int f = \lim \int f_n. \)

**Monotone Convergence Theorem** If \( f_n \geq 0 \) and \( f_n \uparrow f \) pointwise, then \( \int f_n \uparrow \int f. \) (Note: this is true even when the integrals diverge.)

**Uniform Convergence Theorem** If \( f_n \to f \) uniformly, and \( f_n, f \) are integrable, then \( \lim \int f_n = \int f. \)

**Fatou’s Lemma** \( \int \lim \inf f_n \leq \lim \inf \int f_n \) (for \( f_n \geq 0 \)).

Theorems about the convergence of improper integrals.

**Abel** \( f(t), g(t) \) integrable in \([a, b] \forall b > a\), \( g(t) \) monotone and bounded in \([a, \infty)\), \( \int_a^\infty f(t)\,dt \) converges \( \Rightarrow \int_a^\infty f(t)g(t)\,dt \) converges.

**Dirichlet** \( f, g, g’ \) continuous on \([a, \infty)\), \( \lim_{x \to \infty} g(x) = 0 \), \( \int_a^\infty |g’| \) converges, all of the partial integrals of \( f(t) \) are bounded \( (\int_a^b f(t)\,dt) < M \forall b \) \( \Rightarrow \int_a^\infty f(t)g(t)\,dt \) converges.

A useful corollary: \( f(t), g(t) \) integrable in \([a, b] \forall b > a\), \( g(t) \) monotone in \([a, \infty)\) and decreases to 0 as \( t \to \infty \), all of the partial integrals of \( f(t) \) are bounded \( \Rightarrow \int_a^\infty f(t)g(t)\,dt \) converges.