A decision making example using mathematical expectations

A decision regarding how to invest \$100,000 is to be made where two stocks are under consideration, Stock-1 and Stock-2. The following information is given:

with 30% chance, Stock-1 will rise by 20%,
with 70% chance, Stock-1 will decline by 5%,
with 40% chance, Stock-2 will rise by 25%,
with 60% chance, Stock-2 will decline by 10%.

From this data, we see that Stock-2 has the potential of yielding more than Stock-1, but if it happens to decline, it will cause a bigger loss than Stock-1. On the other hand, when we look at the probabilities, the probability of a decline in the value of Stock-1 is relatively larger. How should the decision be made? It depends on what we are after. Let us illustrate this by asking the following questions:

1. Which stock should we invest in if we want to maximize the expected profit?

   Let us denote the possible amounts of gain or loss due to investing the money in Stock-1 or in Stock-2 by G_1, L_1, G_2 and L_2. Then we have

   $$G_1 = 100,000 \times (0.2) = 20,000$$

   $$L_1 = 100,000 \times (-0.05) = -5,000$$

   $$G_2 = 100,000 \times (0.25) = 25,000$$

   $$L_2 = 100,000 \times (-0.10) = -10,000$$

   To find the expectations, we must weight these amounts using the corresponding probabilities. The expected profit E_1 from Stock-1 is then:

   $$E_1 = 20,000 \times (0.3) + (-5,000) \times (0.7) = 2,500.$$

   The expected profit E_2 from Stock-2 is:

   $$E_2 = 25,000 \times (0.4) + (-10,000) \times (0.6) = 4,000.$$

   This says that Stock-2 offers a larger expected profit.

2. Which stock should we invest in if we want to minimize the potential loss?

   Now, the answer is Stock-1. (5,000 vs. 10,000)

3. Which stock should we invest in if we want to maximize the potential gain?

   Now, the answer is again Stock-2. (20,000 vs. 25,000)

4. Suppose we want to split the investments between these two stock by investing S_1 dollars on Stock-1 and S_2 dollars on Stock-2. Is there a way to do this so that the losses will never be more than $7,000 should both stocks decline while keeping the expected profit above $3,000?

   Note that investing the full amount on Stock-1 guarantees that we will never lose more than $5,000 but its expected profit E_1 is less than $3,000. On the other hand investing the full amount on Stock-2 would have $4,000 in expected profit but could potentially yield a $10,000 loss. This is why we consider mixing the two.

   We have the constraint

   $$S_1 + S_2 = 100,000.$$

   We also need to modify G_1, L_1, G_2 and L_2. We now have

   $$G_1 = 0.2S_1, \quad L_1 = -0.05S_1,$$

   so that

   $$E_1 = 0.3G_1 + 0.7L_1 = 0.025S_1.$$

   Similarly, we have

   $$G_2 = 0.25S_2, \quad L_2 = -0.10S_2,$$

   $$E_2 = 0.4G_2 + 0.6L_2 = 0.04S_2.$$

   The total expected profit is $E_1 + E_2$, which we want to keep above $3,000. This translates into

   $$0.025S_1 + 0.04S_2 \geq 3,000.$$

   The maximum possible loss is $|L_1| + |L_2|$, which we want to keep below $7,000$. This translates into

   $$0.05S_1 + 0.10S_2 \leq 7,000.$$

Let us reduce all this into one unknown, say S_1. For this, we solve for S_2 from the equality constraint and find that $S_2 = 100,000 - S_1$. Now, the first inequality objective becomes

$$0.025S_1 + 0.04(100,000 - S_1) \geq 3,000,$$

or if we simplify,

$$0.015S_1 \leq 1,000 \quad \Rightarrow \quad S_1 \leq 66,667.$$

The second objective now says that

$$0.05S_1 + 0.10(100,000 - S_1) \leq 7,000,$$

or if we simplify,

$$0.05S_1 \geq 3,000 \quad \Rightarrow \quad S_1 \geq 60,000.$$

We find from these two constraints that we must at least invest \$60,000 on S_1, but not more than \$66,667.

Exercises:

- Do we still have a mixed investment strategy if instead we want to keep the worst case loss below \$6,000 with the same minimum expected profit amount of \$3,000?

- Find the investment strategy (i.e, the values of S_1 and S_2) that yields the maximum possible expected profit under the original constraint that the worst case loss \$7,000. (This is slightly more difficult.)