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References


AN ANCIENT GREEK APPROXIMATION OF $\sqrt{2}$

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It is was in the Pythagorean School in Croton, Southern Italy, in the 5th century BC that the first demonstration of the irrationality of $\sqrt{2}$ was given. The motivation wasn’t algebraic but rather geometric. Whereas nowadays we describe irrationality by saying that it is impossible to represent $\sqrt{2}$ as a fraction – that is, to find integers $x$ and $y$ such that $x/y = \sqrt{2}$ – the Greek understanding was that the hypotenuse of a right angle triangle with equal sides couldn’t be put into a whole number ratio with one of those sides. The reaction to the proof, which was not publicised due to the Pythagorean culture of secrecy, is legendary. One retelling claims that the person who proved the theorem was shipwrecked at sea as punishment from the gods! For the Pythagoreans, it was perhaps more than just a mathematical fact. They had developed a system of belief - mathematical and religious - that rested on the notion of a universe created out of “perfect” whole numbers, and the theorem shattered that. One might argue that the mathematician involved didn’t so much prove a theorem as demonstrate the non-existence of a certain kind of god.

Pre-Greek civilisations had approximations of $\sqrt{2}$ for many hundreds of years before Pythagoras, but after the irrationality proof these approximations became the only thing to aim for: an exact value was, of course, impossible. The Greeks
found many means to approximate its value, and we will look at one in detail in this article. This method involves constructing two columns of numbers, each starting with a 1. To get the next element in the left column one adds the two elements in the previous row; to get the next element in the right column one adds twice the left element of the previous row to the right element. We can write this mapping algebraically as \((a, b) \rightarrow (a + b, 2a + b)\). The first few \((a, b)\) pairs are then

\[
(1, 1) \quad (2, 3) \quad (5, 7) \quad (12, 17) \quad (29, 41) \quad (70, 99)
\]

It was observed that when one divided one of the \(b\)s by its corresponding \(a\) that the result got closer and closer to \(\sqrt{2}\) the further down the list one went. Bearing in mind that \(\sqrt{2} = 1.414213 \ldots\) the approximations are

\[
\frac{17}{12} = 1.41666 \ldots \quad \frac{41}{29} = 1.413793 \ldots \quad \frac{99}{70} = 1.414285 \ldots
\]

The last approximation here is correct to 4 decimal places. The first was actually known to Babylonian mathematicians. Of course, it's not good enough to merely take the series at face value and assume it approaches \(\sqrt{2}\) : to do so would be against the mathematical culture of rigour that has its roots in the Pythagorean school! It's not inconceivable that the series would be converging to some other number very near \(\sqrt{2}\). Then, if we can prove that it’s approaching \(\sqrt{2}\) we may ask how fast it is doing it. Or, to put it in more modern terms, if I want the first 40 digits of \(\sqrt{2}\), how far down the series do I need to go?

### 1. An approximating method

To answer these questions we return to our hypothetical representation of \(\sqrt{2}\) as a rational number, \(b/a = \sqrt{2}\), which is equivalent to

\[
b^2 - 2a^2 = 0 \tag{9}
\]

As we are only interested in integer solutions we call this an integral or Diophantine equation. The irrationality proof says that we cannot find integers \(a\) and \(b\) to satisfy it. So, in order to find some \(a\) and \(b\) that approximate \(\sqrt{2}\) we want to amend (1) slightly in the hope that the solutions to the similar amended equation will yield approximations. As the equation (1) is integral, the smallest “error” we can introduce is \(\pm 1\). Adding this gives a new equation

\[
b^2 - 2a^2 = \pm 1 \tag{10}
\]
So far, a natural development. If we solve it for \( b/a \), have

\[
\frac{b}{a} = \sqrt{2 \pm \frac{1}{a^2}} \tag{11}
\]

and we see that if \( a \) gets very large then \( 1/a^2 \) gets very small so that quotient \( b/a \) gets closer and closer to \( \sqrt{2} \).

Next, we observe that the first few \((a, b)\) number pairs actually satisfy (2), for

\[
(1)^2 - 2(1)^2 = -1 \quad (3)^2 - 2(2)^2 = +1 \quad (7)^2 - 2(5)^2 = -1
\]

and we notice that the sign of the right hand side alternates. Do all the elements of the \((a, b)\) series satisfy the equation, or is it just the first few? For ease of discussion we introduce the notation \((a_n, b_n)\) for the \(n\)th element of the series of pairs. If it is the case that all the \((a_n, b_n)\) elements satisfy (2) then we will have

\[
\lim_{n \to \infty} \frac{b_n}{a_n} = \sqrt{2}
\]

The following proposition says that this is indeed the case.

**Proposition 1.** Given the series \((a_n, b_n)\) such that \((a_{n+1}, b_{n+1}) = (a_n + b_n, 2a_n + b_n)\) and \((a_1, b_1) = (1, 1)\), we have \(b_n^2 - 2a_n^2 = (-1)^n\).

**Proof.** The series is defined recursively so an inductive proof is called for. We have demonstrated the \(n = 1\) case already. Now suppose the proposition holds from some \(n = k\) so that

\[
b_k^2 - 2a_k^2 = (-1)^k \tag{12}
\]

We have

\[
a_{k+1} = a_k + b_k \quad b_{k+1} = 2a_k + b_k
\]

Solving for \(a_k\) and \(b_k\) we get

\[
a_k = b_{k+1} - a_{k+1} \quad b_k = 2a_{k+1} - b_{k+1} \tag{13}
\]

The proof follows by substitution of (5) into (4):

\[
(2a_{k+1} - b_{k+1})^2 - 2(b_{k+1} - a_{k+1})^2 = (-1)^k
\]

\[
2a_{k+1}^2 - b_{k+1}^2 = (-1)^k
\]

\[
b_{k+1}^2 - 2a_{k+1}^2 = (-1)^{k+1}
\]

Every element of the series thus satisfies (2). \(\square\)
So then our first question is in the affirmative: the series is an approximation to $\sqrt{2}$, and it gets better for larger $n$. We now wish to quantify that. In particular, if we want our approximation a certain tolerance away from the true value, how far down the series should we go?

2. The first error problem

For this question, we have an error or tolerance value $\xi$, and we wish to find an approximation of $\sqrt{2}$ using our series that is within $\xi$ of it. The error is the absolute value of the difference between the approximation (3) and the actual value; i.e.,

$$\xi_n = \left| 2 + \frac{(-1)^n}{a_n^2} - \sqrt{2} \right|$$

Now, if $n$ is even then $b_n/a_n$ over approximates $\sqrt{2}$ (as $(1)^n$ is positive) and the absolute value is not necessary as the inside is positive. On the other hand, if $n$ is odd $b_n/a_n$ under approximates $\sqrt{2}$, so to take the absolute value we need to multiply the inside expression by (-1). We can then let a $(1)^n$ do the work of the absolute value and write the error as

$$\xi_n = (-1)^n \left( 2 + \frac{(-1)^n}{a_n^2} - \sqrt{2} \right)$$

The question is: how large does $a_n$ need to be to achieve a certain $\xi_n$? To answer this we solve for $a_n$, to give

$$a_n^2 = \frac{1}{2\sqrt{2}\xi_n + (-1)^n\xi_n^2} \leq \frac{1}{2\sqrt{2}\xi_n - \xi_n^2} \quad (14)$$

This expression says that given a $\xi$ if we make $a_n$ as large as $1/(2\sqrt{2}\xi_n - \xi_n^2)$ then the error in the resulting approximation will be less than $\xi$. But how big does $n$ need to be to get $a_n$ this large? We have the inequality

$$\frac{(1 + \sqrt{2})^n + 1}{2\sqrt{2}} \leq a_n \quad (15)$$

The proof of this is left as an exercise to the reader! The approach is to solve the $(a_n, b_n)$ series for a series in $a_n$ alone and then apply standard recursion solution methods. This approach also yields another proof of proposition 1.

Now, you may object and say “the goal of this is to find an approximation to $\sqrt{2}$, but in these inequalities you are using the value of $\sqrt{2}$ to find that
approximation! This is true, but (7) is just an inequality and can be readily amended with known numerical values to make the method right. We just make the $\sqrt{2}$ in the numerator slightly smaller and the $\sqrt{2}$ in the denominator slightly bigger to give

$$\frac{(2.4)^n + 1}{2.84} < \frac{(1 + \sqrt{2})^n + 1}{2\sqrt{2}} \leq a_n$$

However, for ease of exposition we’ll leave the $\sqrt{2}$ in and agree to amend the expression appropriately later.

Combining (6) and (7) gives

$$\frac{(1 + \sqrt{2})^n + 1}{2\sqrt{2}} \leq \frac{1}{2\sqrt{2}\xi_n - \xi_n^2}$$

We finally use the logarithmic function to find a rather complicated expression for $n$:

$$n < \log \left( \frac{2\sqrt{2}}{2\sqrt{2}\xi_n - \xi_n^2} - 1 \right) \left( \frac{1}{\log(1 + \sqrt{2})} \right)$$

$$n < \log \left( \frac{1.42}{1.41\xi_n - \xi_n^2} - 1 \right) \quad (1.14)$$

So now, given a $\xi$ we choose $N$ to be at least as large as the right hand side. Then, when we get the pair $(a_N, b_N)$ we are guaranteed that the quotient $b_N/a_N$ will be no further than $\xi$ away from $\sqrt{2}$.

### 3. The second error problem

The attempt to find a close approximation to a given irrational hasn’t captured the modern imagination so much as the search for more and more digits of an irrational. At the time of writing $\sqrt{2}$ has been calculated to 1 trillion decimal places! It worth reminding ourselves that the digits are defined to be the numbers after the decimal place, so the starting 1 doesn’t count. We might think that the previous section provides sufficient machinery to tackle the digits problem, at least in theory. For instance, if we wanted the first 4 digits we might think of setting $\xi = 0.00001 = 0.1 \times 10^{-4}$ and solving appropriately, thinking that the error could only affect the digits beyond the fourth decimal place. But this won’t always hold. For suppose the first 5 digits of 2 were 1.41420 (they aren’t, but a 0 will eventually occur). Then the number 1.41419 is within $10^{-4}$ but is only correct to three decimal places. Okay, you say, set $\xi = 0.1 \times 10^{-5}$ to overcome this. But there could be another 0, and then 1.414199 would again be within
the tolerance while still having fewer digits than desired. In fact, there’s strong
evidence to suggest that the decimal expansion of \( \sqrt{2} \) contains arbitrarily long
strings of 0s, and so no matter how small we set the exponent of 10 there’s still
a chance we’ll be caught out by one of these strings.\(^2\)

We then require a new approach, and our series furnishes one quite nicely. Recall that our approximation is

\[
\frac{b_n}{a_n} = \sqrt{2 + \frac{(-1)^n}{a_n}}
\]

and that, by the expression, this is an over approximation for \( n \) even and an
under approximation for \( n \) odd. Set \( n = 6 \). Our two approximations are

\[
\frac{b_6}{a_6} = \frac{99}{90} = 1.414285 \ldots \quad \frac{b_7}{a_7} = \frac{239}{169} = 1.414201 \ldots
\]

These share 4 digits in common. Now, any number between these two numbers
has to have the same first 4 first digits, for otherwise it would be either bigger
or smaller than one of the numbers. But \( \sqrt{2} \) is between these numbers! So we
can say with absolute certainty that the first 4 digits of 2 are given by 1.4142!
For the general problem, suppose we want \( k \) digits of 2. We set \( \xi = 0.1 \times 10^{-k} \),
determine \( n \) and then \( (a_{n+1}, b_{n+1}) \). We get the all the first digits that \( b_n/a_n \) and
\( b_{n+1}/a_{n+1} \) have in common. If there are \( k \) of these then we are done; otherwise,
we just increase \( n \) by 1, test again, and continue until we have the required \( k \)
digits.

Notes

1 by Shigeru Kondo and Alexander Yee; see http://www.numberworld.org/. The same
authors have calculated the first 10 trillion digits of \( \pi \), and the software they did this on
is freely downloadable.

2 The evidence here is the proof that almost all number are “normal”: in any base
each “digit”, in that base, will appear with the same frequency in the expansion. In base
10, \( n \) 0s in a row is a digit and so will appear at least sometime in all normal numbers.
Though there is no proof of the normality of \( \sqrt{2} \), empirical evidence suggests that it is
normal, as does the “almost all” statement in the normality theorem. Thus we should
expect arbitrary large strings of 0s to occur. For more information on this see Chapters 4
and 9 of [1].

References

[1] G.H. Hardy and E.M. Wright. An Introduction to the Theory of Numbers. Ox-
Alan Turing was a British Mathematician who played a significant role in Bletchley Park as a cryptographic expert during World War Two. Bletchley Park represented a contingent of the finest minds in Britain at the time, all working together to crack the Enigma – the encrypting machine that gave Germany the upper hand in encoded communication during the war. The Enigma machine is a remarkable feat of engineering, similar in appearance to a typewriter. In practice, the user types in a message letter by letter, with corresponding bulbs lighting up displaying the corresponding encoded letter. Turing’s ingenuity was pivotal in solving the seemingly insurmountable problem of cracking the Enigma, and similar creativity and talent is evident in his paper *On Computable Numbers with an Application to the Entscheidungsproblem*. The *Entscheidungsproblem* (German for Decision Problem) represented one of the stiffest mathematical problems at the time. Its origins are archaic, possibly dating back even to Leibniz, but it was the German mathematician David Hilbert who was most renowned for proposing it:

Find an algorithmic procedure that decides the validity of any given mathematical statement.

Of course, should such an algorithm exist, it would be of unparalleled practicality. For it could, for instance, prove or disprove the validity of the Riemann Hypothesis, and we would not even necessarily need to know the reasons for its truth or falsity. Alan Turing eventually solved this problem, using a unique and interesting approach. Firstly, central to Turing’s proof is the idea of a Turing Machine. A Turing Machine is akin to a computer in that it takes a given input and processes it to produce an output. In fact, one could even argue that computers themselves were modelled on the idea of Turing Machines. A Turing Machine is quite primitive: it receives instructions on a piece of tape which passes through it, which the Turing Machine manipulates according to certain rules. Although Turing Machines are more theoretical than practical, one could imagine they would look something like figure 1.