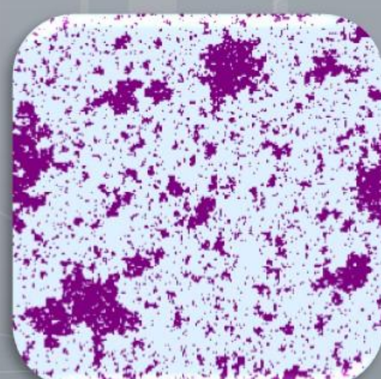
Stanford University

Math Colloquium

May 2012

The Static and Stochastic Ising Models



Eyal Lubetzky

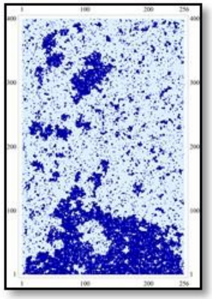
Microsoft Research

The Ising model

Wilhelm Lenz
1888–1957

- Introduced by Wilhelm Lenz in 1920 as a model of *ferromagnetism*:
  - Place iron in a magnetic field: increase field to maximum , then slowly reduce it to zero.
  - There is a critical temperature T_c (the Curie point) below which the iron retains residual magnetism.
- Magnetism caused by charged particles spinning or moving in orbit in alignment with each other.
- How do local interactions between nearby particles affect the global behavior at different temperatures?

The Ising model



- Gives random binary values (spins) to vertices accounting for nearest-neighbor interactions.
- Initially thought to be over-simplified to capture ferromagnetism.

Sociology
Biology
Chemistry
Economics
Physics
…

- Turned out to have a crucial role in the understanding of phase transitions and critical phenomena.
- One of the most studied models in Math. Phys.: more than 10,000 papers over the last 25 years…

Google scholar allintitle: ising

Scholar Articles excluding patents 1985 - 2010

Results **1** - **10** of about **10,800**.

Ordered phase of short-range **Ising** spin-glasses
DS Fisher, DA Huse - Physical Review Letters, 1986 - APS
We propose a new picture of the **Ising**-spin-glass phase, based on an Ansatz for the scaling of
low-lying large-scale-droplet excitations. We find behavior very different from the infinite-range
model. The truncated spatial correlations decay as a power of distance, the ac nonlinear **...**
Cited by 375 - Related articles - All 3 versions

bing ising model

ALL RESULTS 1-10 of 150,000 results

Ising model - Wikipedia, the free encyclopedia
Definition · General discussion · Historical significance · Applications
The **Ising model** is a mathematical **model** of ferromagnetism in statistical mechanics.
consists of discrete variables called spins that can be in one of two states.

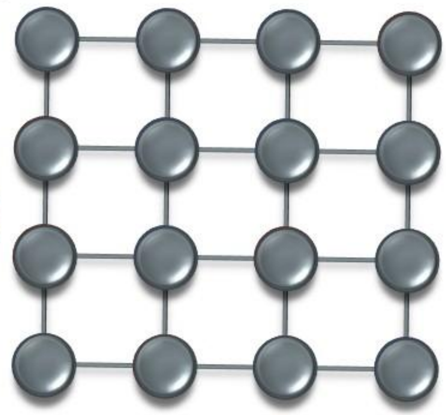
Definition: the classical Ising model

- Underlying geometry: Λ = finite 2D grid.
- Set of possible configurations:

$$\Omega = \{\pm 1\}^{\Lambda}$$

(each *site* receives a plus/minus *spin*)

- Probability of a configuration $\sigma \in \Omega$ given by the *Gibbs distribution*:



$$\mu(\sigma) = \frac{1}{Z(\beta)} \exp\Big(\beta \sum_{x\sim y} \sigma(x)\sigma(y)\Big)$$

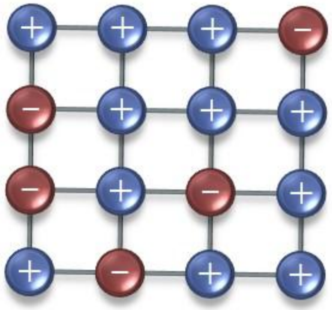
Partition function

Inverse temperature $\beta \geq 0$

Zero external field

The classical Ising model

- $\mu(\sigma) \propto \exp\Big(\beta \sum_{x\sim y} \sigma(x)\sigma(y)\Big)$ for $\sigma \in \Omega = \{\pm 1\}^{\Lambda}$
  - Larger β favors configurations with aligned spins at neighboring sites.
  - Spin interactions ≈ local, justified by the rapid decay of magnetic force with distance.



- The *magnetization* is the (normalized) sum of spins:
$$M(\sigma) = \left|\Lambda\right|^{-1} \sum_{x\in\Lambda} \sigma(x)$$
  - Distinguishes between disorder ($M \approx 0$) and order.

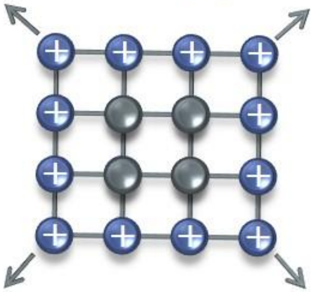
The magnetization (sum of spins)

- Naturally corresponds to quantities exhibiting phase-transitions in various systems, e.g.:
  - Binary alloys (spins denote the molecule type)
  - Lattice-gas (spins denote matter/holes).
- By symmetry $\mathbb{E}[M(\sigma)] = 0$ [recall $\mu(\sigma) \propto \exp\Big(\beta \sum_{x\sim y} \sigma(x)\sigma(y)\Big)$].
- What if we *break the symmetry* by forcing some ⊕'s? How do we then calculate the expected $M(\sigma)$?
  - Even for a tiny 10×10 lattice the normalizer $Z(\beta)$ is already a sum over $2^{|\Lambda|} = 2^{100}$ terms…

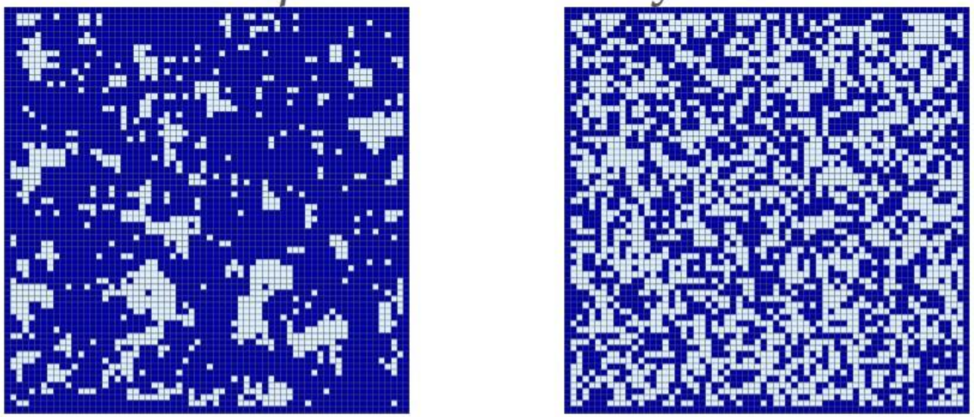
The Ising phase-transition

- Ferromagnetism in this setting: [recall $M(\sigma) = \frac{1}{|\Lambda|}\sum \sigma(x)$]
  - Condition on the boundary sites all having *plus* spins.
  - Let the system size $|\Lambda|$ tend $\to \infty$

  ($\approx$ a magnetic field with effect $\to 0$).

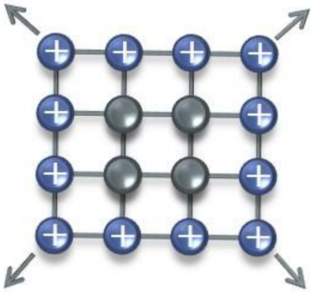


- What is the typical $M(\sigma)$ for large $|\Lambda|$?
  Does the effect of *plus* boundary vanish in the limit?



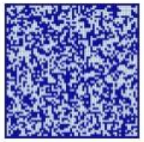
The Ising phase-transition (ctd.)

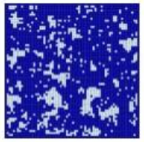
- Ferromagnetism in this setting: [recall $M(\sigma) = \frac{1}{|\Lambda|}\sum \sigma(x)$]
  - Condition on the boundary sites all having *plus* spins.
  - Let the system size $|\Lambda|$ tend $\to \infty$



- Expect: phase-transition at some critical β_c :

$$\lim_{|\Lambda|\to\infty} \mathbb{E}^+[M(\sigma)] = \begin{cases} 0 & \text{if } \beta < \beta_c \\ c_\beta > 0 & \text{if } \beta > \beta_c \end{cases}$$



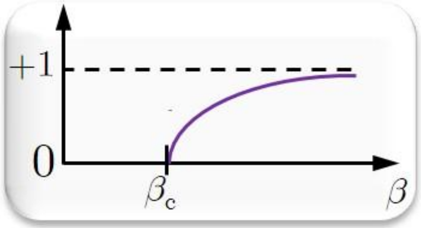


all-plus boundary

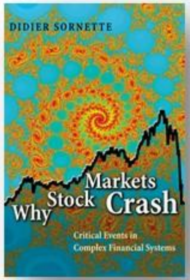
spontaneous magnetization

The Ising phase-transition (ctd.)

- The magnetization phase-transition at β_c :



- Replace *magnetization* ⟷ *price* to find diagram in "Why Stock Markets Crash" / D. Sornette (2001) [Chapter 5 "Modeling bubbles and crashes"]



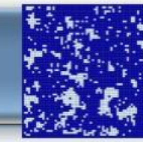

D. Sornette

- Such applications of the Ising Model emphasize a missing dimension of ***time***:
  - How does the system evolve?
  - From a given starting state, how long does it take for certain configurations to appear?

Static vs. stochastic Ising

- Expected behavior for the Ising distribution:

$\beta < \beta_c$	β_c	$\beta > \beta_c$
$\mathbb{E}^+[M(\sigma)] \xrightarrow[\lvert\Lambda\rvert\to\infty]{} 0$		$\mathbb{E}^+[M(\sigma)] \xrightarrow[\lvert\Lambda\rvert\to\infty]{} c_\beta > 0$

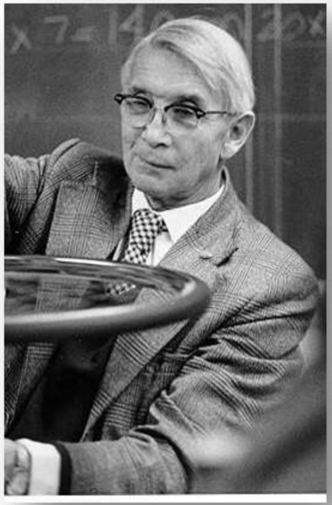


- Expected behavior for rate of convergence of dynamics:

$\beta < \beta_c$	β_c	$\beta > \beta_c$
constant	power law	exponentially slow — ***Free** b.c.*
		power law — ***Plus** b.c.*

  - Symmetry breaking:
- More on this later…

The 1D Ising model



Ernst Ising
1900-1998

- Ph.D. in Physics in 1924 from U. Hamburg under the supervision of Lenz.
- Studied the 1D model of Lenz in his thesis:

  Beitrag zur theorie des ferromagnetismus
  E **Ising** - Zeitschrift für Physik A Hadrons and Nuclei, 1925 - Springer
  Cited by 1043

  - Gave an exact solution for the 1D model.
  - Unfortunately: no phase-transition…
  - Gave heuristic arguments why there would not be a phase-transition in higher dimensions either (calling for more sophisticated models of ferromagnetism).

The 1D Ising

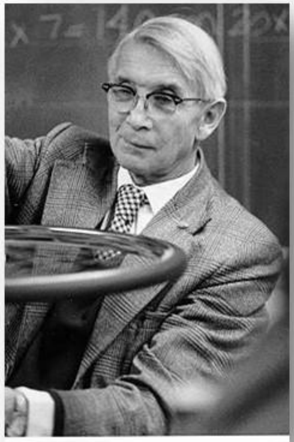
- Setting: Λ = vertices along a line.

- An instance of the Ising model can be generated by running a 2-state Markov chain:
  - $K \propto \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}$ indexed by the states $\{\pm 1\}$. ← *transfer matrix*
- Intuition for absence of a phase-transition: No consideration for history beyond last seen vertex, e.g. (

) similar to (

).
- $\lim_{|\Lambda|\to\infty} \mathbb{E}^{+}[M(\sigma)] = 0$ for any $\beta > 0$.

After solving the 1D model



- Ising [letter to S. Brush in 1967]:
  "...I discussed the result of my paper widely with Prof. Lenz and with Dr. Wolfgang Pauli, who at that time was teaching in Hamburg. There was some disappointment that the linear model did not show the expected ferromagnetic properties..."
- Left research after a few years at the German General Electric Co. and turned to teaching in public schools.
- Survived WW2 in Luxembourg isolated from scientific life. Came to the US in 1947 and only then "...did I learn that the idea had been expanded."

Meanwhile, on 2D Ising

- Heisenberg (1928) proposed his own theory of ferromagnetism, with Ising's negative result as a motivation for the more sophisticated model.
- Followed by other models attempting to explain order/disorder in metallic alloys.
- In 1936 Rudolf Peierls published the paper

On **Ising's model** of **ferromagnetism**
R Peierls - Mathematical Proceedings of the Cambridge ..., 1936
Ising* discussed the following **model** of a **ferromagnetic** body:
of moment yn to be arranged in a regular lattice; each of them is
orientations, which we call positive and negative. Assume further t
Cited by 327 - Related articles - All 3 versions

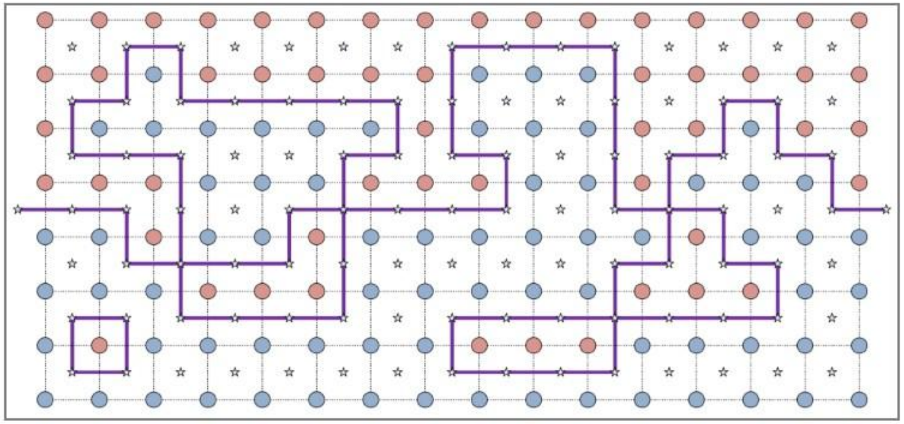
arguing that the 2D and 3D Ising models *do* have spontaneous magnetization at *low enough temperature* (contrary to Ising's prediction).

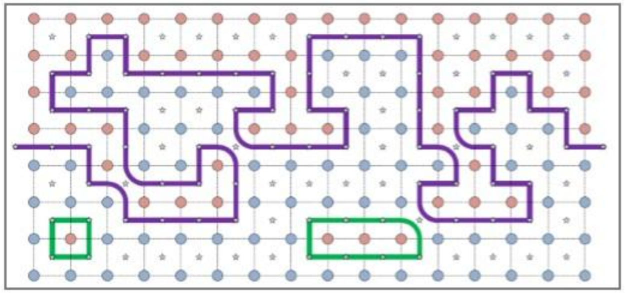
W. Heisenberg
1901-1976

R. Peierls
1907-1995

Peierls' phase transition argument

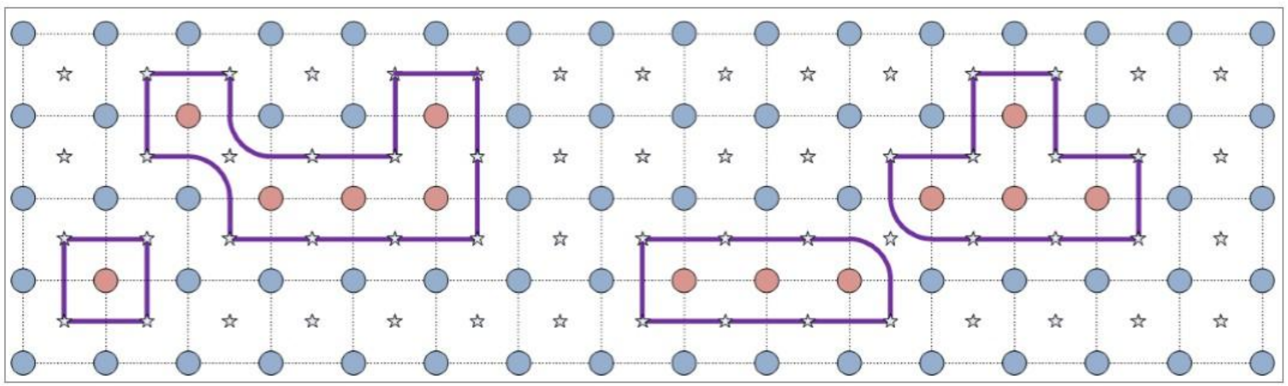
- Peierls' combinatorial argument is simple and robust.
- Key idea: represent Ising configurations as *contours* in the *dual graph*: the edges are dual to disagreeing edges.





Peierls' phase transition argument

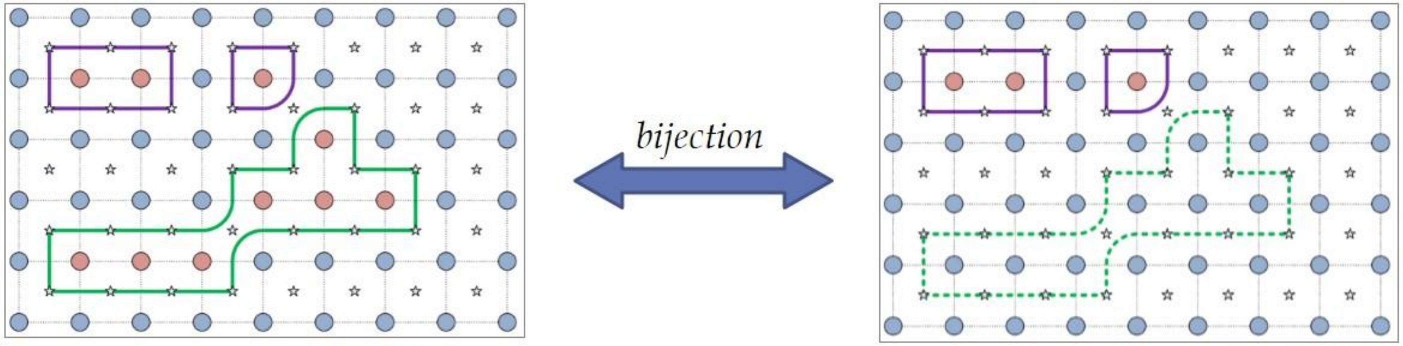
- When all boundary spins are + 's the Peierls contours are all closed [marking "islands" containing of − 's].



- The proof will follow a first moment argument on the number of sites inside such a − component.

Peierls' phase transition argument

- Setting: $\Lambda \subset \mathbb{Z}^2$ is an $n \times n$ box with all-plus boundary.
- Fix a contour C of length ℓ.
- For each σ containing C flip all the spins of C *and its interior* to arrive at a unique σ' :



$\mu^+(\sigma) = Z^{-1} e^{B-\beta\ell}$ $\mu^+(\sigma') = Z^{-1} e^{B+\beta\ell}$

- $\Rightarrow \mathbb{P}(C$ belongs to configuration contours$) \le e^{-2\beta\ell}$.

Peierls' phase transition argument

- For a fixed contour C of length ℓ :
  - $\mathbb{P}(C \text{ belongs to contours}) \le e^{-2\beta\ell}$.
  - C can contain at most ℓ^2 sites (isoperimetric).
- At most $4N \cdot 3^{\ell-1}$ possible such contours, where $N = n^2$ is the total number of sites.
- Summing we get:

$$\mathbb{E}[\#\{i : \sigma(i) = -1\}] \le \tfrac{4}{3} N \sum_\ell \ell^2 (3e^{-2\beta})^\ell < \varepsilon N$$

where $\varepsilon < ½$ for a suitably large β . ■

Exact solution of 2D Ising

- Critical point candidate $\beta_c = \frac{1}{2}\log(1+\sqrt{2}) \approx 0.44$ found by Kramers and Wannier in 1941 via duality.

- 2D Ising model was *exactly solved* in 1944 in the seminal work of Lars Onsager (Nobel in Chemistry 1968)
  - Proof used the transfer matrix method.
  - For the 2D lattice the transfer matrix ($2^n \times 2^n$) was analyzed using the theory of Lie algebras.

L. Onsager
1903-1976

Magnetization at low temperature

- What is the spontaneous magnetization at low temp?
  - Onsager wrote the solution on the blackboard at Cornell in 1948:

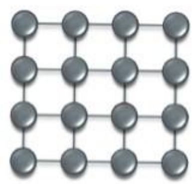
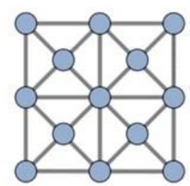
$$\left[1-\left(\sinh(2\beta)\right)^{-4}\right]^{1/8}$$

    Appeared later in print as a remark without proof.
  - In 1952, C.N. Yang (Nobel in Physics 1957) succeeded in re-deriving Onsager's formula.

C.N. Yang

- What happens when $\beta \to \beta_c$ from above?
  - The spontaneous magnetization is $\asymp (\beta-\beta_c)^{1/8}$. This is an example of a *critical exponent*.

Universality and critical exponents

- Despite simplified description, physicists believe (supported by many experiments) that the Ising model shares many critical phenomena with various other (far more complex) systems: a *universality class*.
- Some properties (e.g. the value of β_c) can certainly depend on the model, e.g. the underlying lattice type:
  - lattice:   
  - β_c : $\frac{1}{2}\log(1+\sqrt{2})$ $\frac{1}{4}\log(3)$ $\frac{1}{2}\log(2+\sqrt{3})$

while scaling limit and critical exponents are the same.

Example: critical Ising interfaces

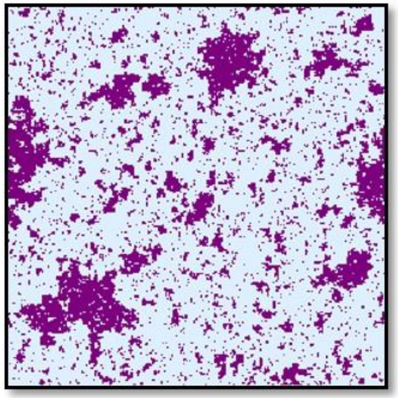
- Understanding of limiting geometry boosted by the advent of SLE [Schramm ’00], CLE and tools to study conformally invariant systems.

O. Schramm
1961-2008

- Recent breakthrough results due to Stas Smirnov describe the full scaling limit of the 2D Ising cluster interfaces.
  - Fields medal citation read: ”for the proof of conformal invariance of percolation and the planar Ising model in statistical physics”.

S. Smirnov

- Interfaces between the +/- components have a conformally invariant limit: SLE(3) (regardless of the 2D lattice type).
- Static Ising properties are a prerequisite for understanding its dynamics.



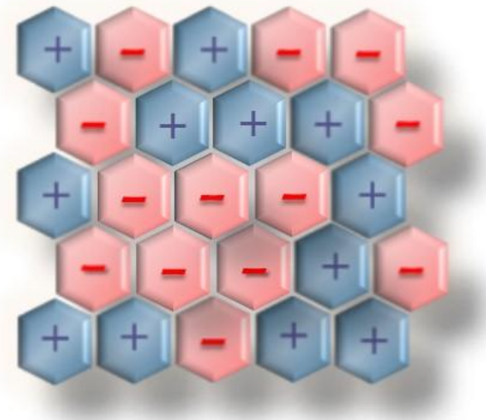
Glauber dynamics / Stochastic Ising


R.J. Glauber

- Glauber dynamics for the Ising model (also known as the *Stochastic Ising model*) introduced in 1963 by Roy J. Glauber (Nobel in Physics 2005).
  - finite ergodic Markov chain on $\Omega = \{\pm 1\}^{\Lambda}$
  - moves between states by flipping a single site.
  - converges to the stationary Ising measure μ.
- Intensively studied over the last 30 years:
  - Natural efficient sampler for the Ising model.
  - Captures its stochastic evolution.

Glauber dynamics for Ising

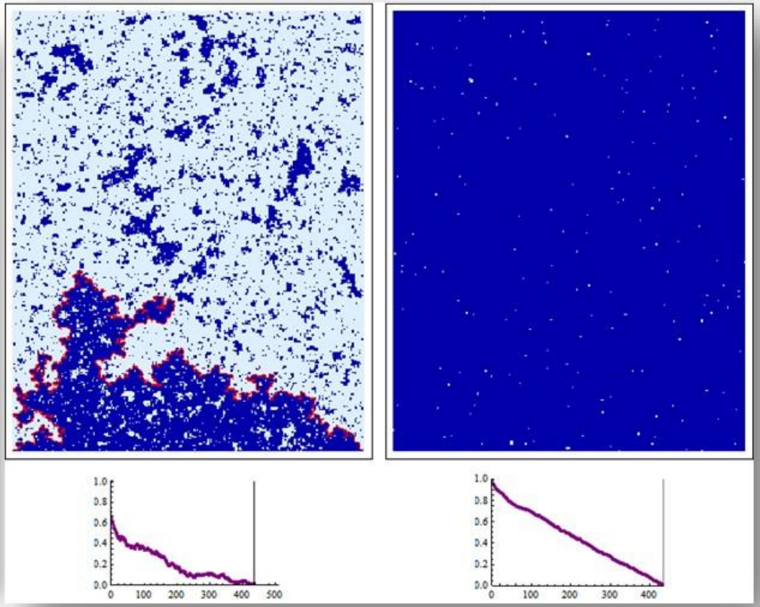
- One of the most commonly used MC samplers for the Ising distribution μ :
  - Update sites via *iid* Poisson(1) clocks
  - Each update replaces a spin at $u \in V$ by a new spin $\sim \mu$ conditioned on all remaining spins at $V \setminus \{u\}$.



- The above is the *heat-bath* version. Other versions of the dynamics include e.g. Metropolis.
- To sample from the Ising model, start at an arbitrary state (e.g. all-plus) run the chain.
  - How long does it take it to converge to μ ?

Example: Glauber dynamics for critical Ising on the square lattice

- 256 x 320 square lattice w. boundary conditions:
  - (+) at bottom
  - (−) elsewhere.
- Frame after 2^{23} steps, i.e. $\sim$100 updates per site.



Notions of convergence to equilibrium

- Spectral gap in the spectrum of the generator:

  gap = smallest positive eigenvalue of the heat-kernel H_t of the dynamics.

- Mixing time : (according to a given metric).
  - Standard choice: L^1 (total-variation) mixing time to within ε is defined as

$$t_{\mathrm{mix}}(\varepsilon) = \inf\left\{t : \max_{\sigma} \left\| H_t(\sigma,\cdot) - \mu \right\|_{\mathrm{TV}} \leq \varepsilon \right\}.$$

where

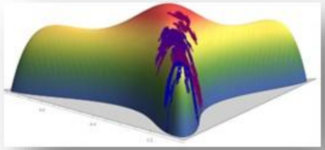
$$\left\| \mu - \nu \right\|_{\mathrm{TV}} = \sup_{A \subset \Omega} \left[\mu(A) - \nu(A) \right].$$

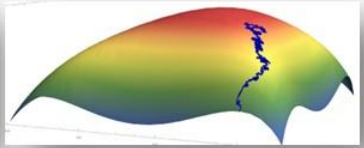
Believed picture for Ising on $\mathbb{Z}_n^d$

- For some critical inverse-temperature β_c:

$\beta < \beta_c$	β_c	$\beta > \beta_c$ (*free* b.c.)
$\text{gap}^{-1} = O(1)$	$\text{gap}^{-1} \asymp n^z$	$\text{gap}^{-1} \asymp e^{(\tau_\beta + o(1)) n^{d-1}}$
$t_{\text{mix}} = O(\log n)$	$t_{\text{mix}} \asymp n^z$ (*universal*)	$t_{\text{mix}} \asymp e^{(\tau_\beta + o(1)) n^{d-1}}$

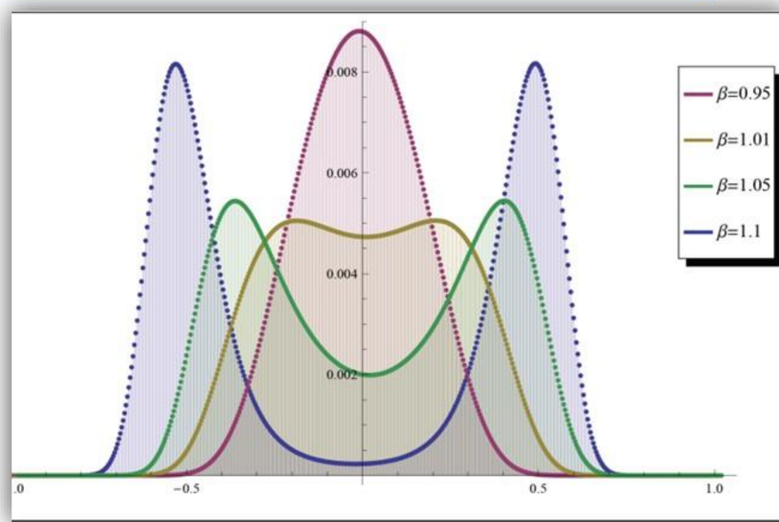
- [Ding, L., Peres ('09a, '09b)] : verified on complete graph **+ *scaling window***
  - E.g.., gap^{-1} is $\frac{1+o(1)}{1-\beta}$ $\quad \asymp n^{1/2}$ $\quad \asymp \frac{1}{\beta-1}\exp[\frac{3}{4}(\beta-1)^2 n]$
- Analogous picture verified for:
  - Regular tree [Berger, Kenyon, Mossel, Peres '05] (high T/low T)
    [Ding, L., Peres '10] (critical T)
  - Potts model on complete graph
    [Cuff, Ding, L., Louidor, Peres, Sly]





Critical slowdown

- **Intuition: low temperature**
  - Exponential mixing due to a bottleneck between the "mostly-plus" and the "mostly-minus" states



Stationary magnetization

- **Intuition: high temperature**
  - At $\beta = 0$ there is complete independence.
  - For very small $\beta > 0$ a spin is likely to choose the same update given 2 very different neighborhoods (weak "communication" between sites).
  - States can be coupled quickly, hence rapid mixing.
- **Intuition: critical power-law**
  - Doubling the box incurs a constant factor in mixing…

Mixing for Ising on the 2D lattice

- Fast mixing at high temperatures:
  - [Aizenman, Holley ’84]
  - [Dobrushin, Shlosman ’87]
  - [Holley, Stroock ’87, ’89]
  - [Holley ’91]
  - [Stroock, Zegarlinski ’92a, ’92b, ’92c]
  - [Zegarlinski ’90, ’92]
  - [Lu, Yau ’93]
  - [Martinelli, Olivieri ’94a, ‘94b]
  - [Martinelli, Olivieri, Schonmann ’94]
- Slow mixing at low temperatures:
  - [Schonmann ’87]
  - [Chayes, Chayes, Schonmann ’87]
  - [Martinelli ’94]
  - [Cesi, Guadagni, Martinelli, Schonmann ’96].

Picture on $\mathbb{Z}^2$

- High & low temperatures fully settled for free b.c.:

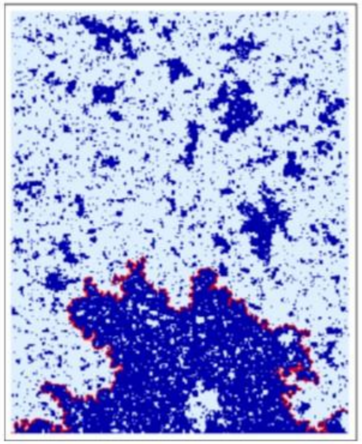
$\beta < \beta_c$ — β_c — $\beta > \beta_c$ *free* b.c.

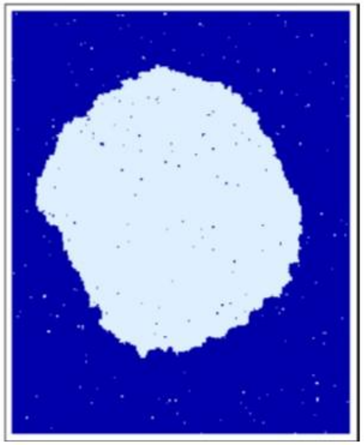
$$\mathrm{gap}^{-1} = O(1)$$
$$t_{\mathrm{mix}} = O(\log n)$$

$$\mathrm{gap}^{-1} = \exp(c(\beta) + o(1))n$$
$$t_{\mathrm{mix}} = \exp(c(\beta) + o(1))n$$

- Unanswered:

Critical power law?



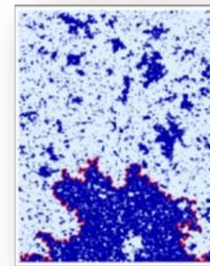


Polynomial mixing under ***plus*** b.c.?

New results: mixing at criticality

β_c

- Prior estimates on critical mixing:
  - Numerical experiments: universal exponent of $\sim \mathbf{2.17}$
    [Ito '93], [Wang *et al* '95], [Grassberger'95], [Nightingale, Blöte'96], [Wang, Hu'97],…
  - [Holley '91]: Mixing is **at least polynomial**.
  - **No sub-exponential upper bounds known.**



- THEOREM: ([L.-Sly '12+])

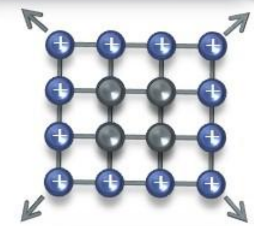
There $\exists$ exists an absolute $c > 0$ so that the mixing time of Glauber dynamics for the critical Ising model on an $n \times n$ box with arbitrary boundary conditions is at most n^c.

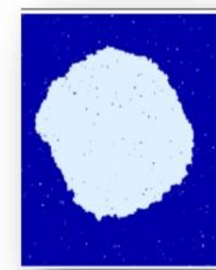
- COROLLARY: Perfect simulation (zero error approximation) for the 2D critical Ising model with arbitrary boundary conditions.

New results: mixing under all–plus

$\beta > \beta_c$ +

- Previous milestones:
  - [Fisher, Huse '87]: conjectured mixing for Ising on an $n \times n$ box with *plus b.c.* is $\asymp n^2$ (Lifshitz's law ('62)).
  - [Martinelli '94]: mixing $\leq \exp\left(n^{1/2+o(1)}\right)$ for large enough β.
  - [Martinelli, Toninelli '10]: $t_{\text{mix}} \leq \exp(n^{\varepsilon})$ for any $\varepsilon > 0$ and large enough β.





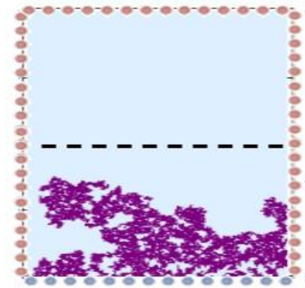
- <u>THEOREM:</u> ([L., Martinelli, Toninelli, Sly '12+]:

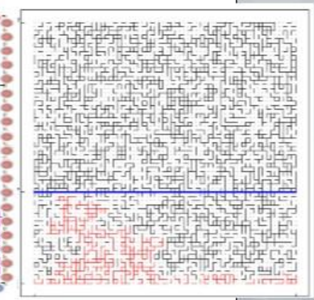
For any $\beta > \beta_c$ there $\exists\, c(\beta) > 0$ so that the mixing time of Glauber dynamics for Ising on an $n \times n$ box with all-plus boundary conditions is at most $n^{c \log n}$.

Quasi-polynomial down to β_c

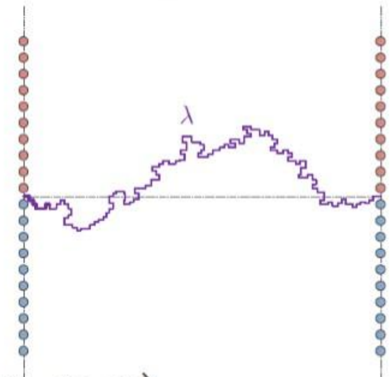
Crucial equilibrium estimates

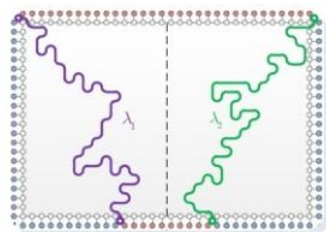
- Analysis of mixing at criticality used SLE behavior of Ising interfaces (RSW-type crossing-probability estimates for FK-Ising by [Chelkak-Smirnov ‘09], [Camia-Newman ‘09], [Duminil-Copin-Hongler-Nolin ‘09])





- Analysis of mixing under all-plus b.c. at low temperatures relied on new quantitative estimates of convergence of Ising interfaces to Brownian bridges (refining [Higuchi ‘79], [Hryniv ‘98], [Greenberg-Ioffe ‘05])





Glauber dynamics on $\mathbb{Z}^2$

- Best known bounds on mixing:

$\beta < \beta_c$

$$\mathrm{gap}^{-1} = O(1)$$

$$t_{\mathrm{mix}} = \tfrac{1}{2}\lambda_\infty^{-1}\log n + O(\log\log n)$$

Cutoff

β_c

$$n^{7/4} \le \mathrm{gap}^{-1} \le t_{\mathrm{mix}} \le n^c$$

$\beta > \beta_c$

Free b.c. :

$$\mathrm{gap}^{-1} = \exp(c(\beta)+o(1))n$$

$$t_{\mathrm{mix}} = \exp(c(\beta)+o(1))n$$

Plus b.c. :

$$\mathrm{gap}^{-1} \le t_{\mathrm{mix}} \le n^{O(\log n)}$$

Open problems

- Calculate the precise dynamical critical exponent.
- Establish power-law behavior on the lattice in 3D.
- Show the Glauber dynamics is polynomial at low temperatures under all-plus boundary conditions.

Thank you