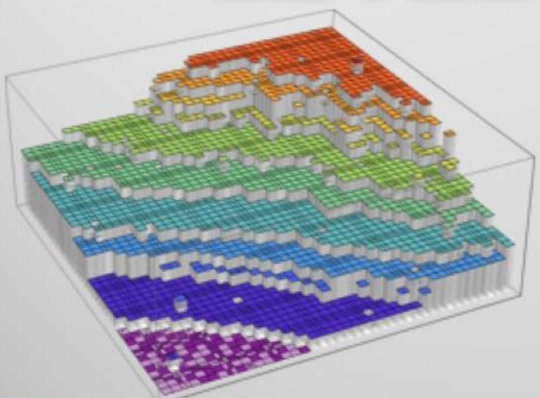
Limiting shape and cube-root fluctuations of the level lines of (2+1)D SOS



Eyal Lubetzky
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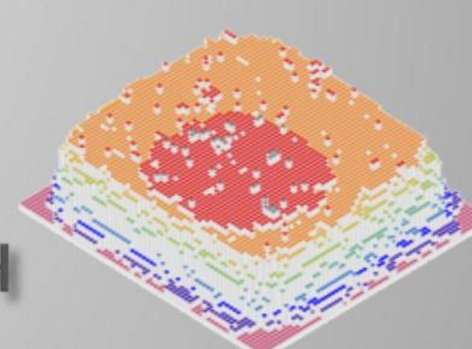


Based on joint works with
P. Caputo, F. Martinelli, A. Sly, and F. Toninelli

Discrete Random Geometry, Aug 2013

The Solid On Solid model

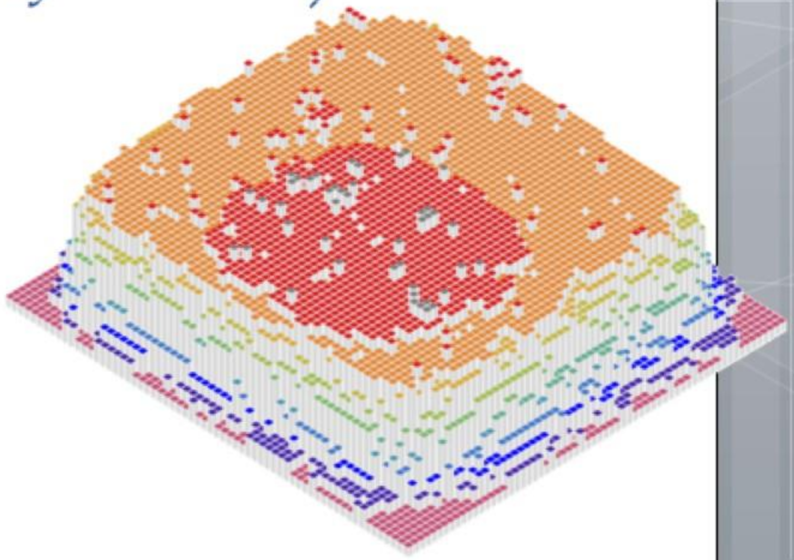
- <u>DEFINITION:</u> *(2+1)-dimensional SOS above a wall*
  [the Onsager-Temperley sheet [Temperley (1952)]]
  probability measure on *height functions* η on $\Lambda = \{1, \dots, L\}^2$ with $\Lambda \ni x \mapsto \eta_x \in \mathbb{Z}_+$ and $\eta_x = 0$ for $x \notin \Lambda$ given by

$$\pi_\Lambda(\eta) = \frac{1}{Z_{\beta,\Lambda}} \exp\left(-\beta \sum_{x\sim y} |\eta_x - \eta_y|\right)$$

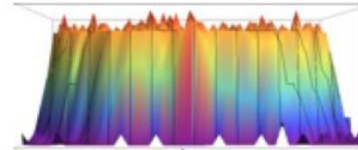
Gibbs distribution

Partition function Z

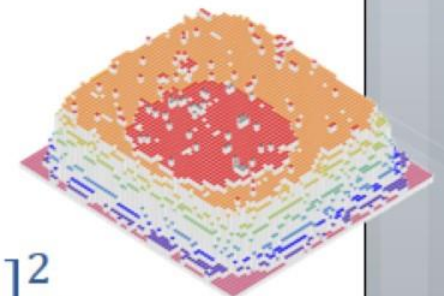
Inverse temperature $\beta \geq 0$



Basic question on (2+1)D SOS



- *Height profile:*
  I. What is the average surface height? (diverges?) How concentrated are the heights? What is the maximum?

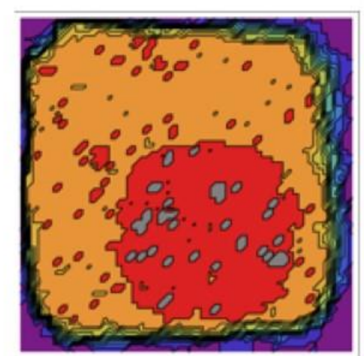


- *Shape:* rescale ensemble of level lines → $[0,1]^2$.
  II. Is there a scaling limit, *e.g.*, in Hausdorff distance?

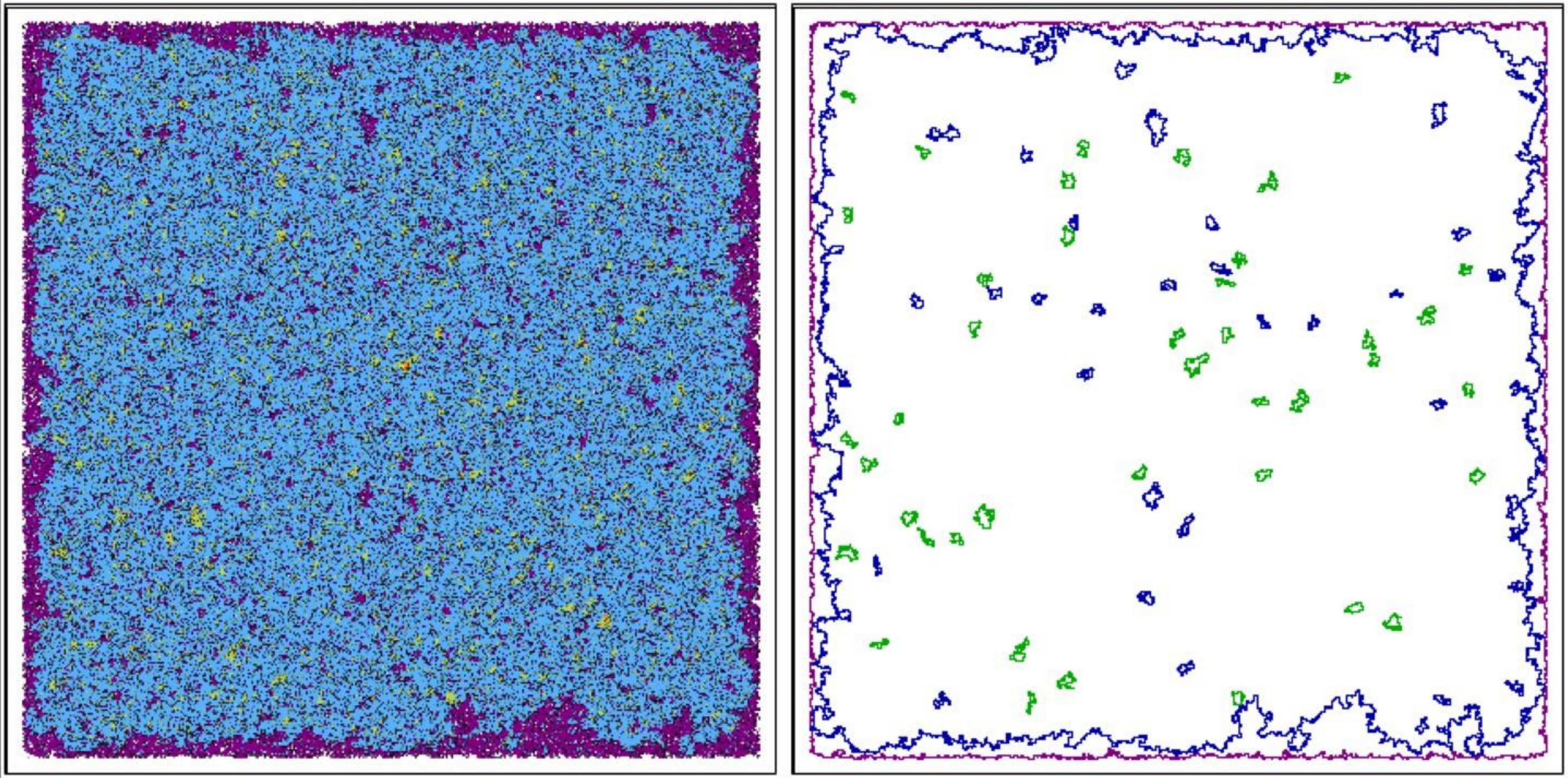
  If so:

  III. Can the limit be explicitly described?

  IV. Quantitative estimates: for finite *L* what are the fluctuations of the level lines around their limit?



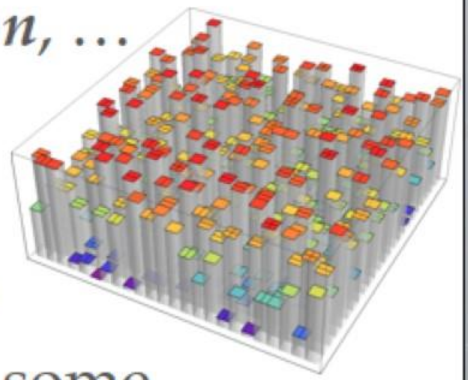
Loop ensemble of SOS level lines



SOS configuration on a 1000×1000 *square.*
Showing loops of length ≥ 100.

The Solid On Solid model (history)

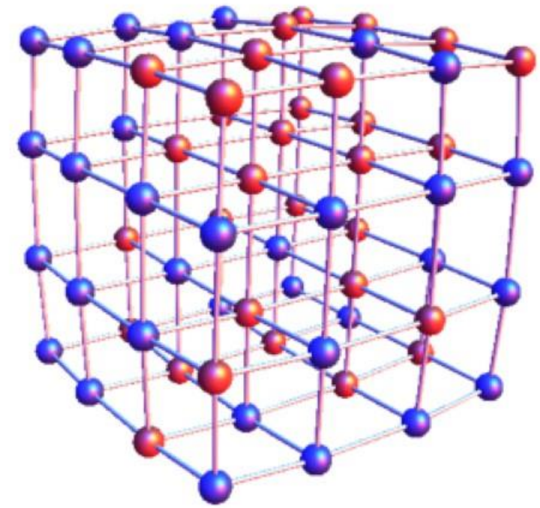
- Introduced in [Temperley ('52)]: (1+1)D w/o walls.
- Extensively studied random surface phenomena: ***roughening transition, wetting transition***, …
- Some of the flavors studied:
  - Attractive potential to the walls.
  - Real valued heights (no temperature).
  - Hamiltonian of $-\beta \sum_{x\sim y}|\eta_x - \eta_y|^p$ for some $p > 0$ (the case $p = 2$ is the *discrete Gaussian model*)
  - *Restricted SOS* (RSOS): only allows gradients $\{0, \pm 1\}$
  - *Body Centered SOS* (BCSOS, [*van Beijeren model* ('77)): two intertwined lattices with hard constraints
  - …



Motivation: 3D Ising interfaces

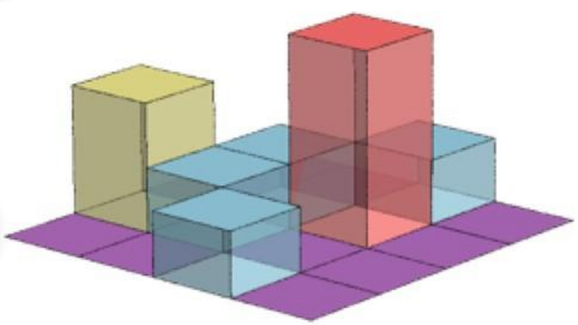
- 3D Ising model:
  - assigns +/− spins to $\Lambda \subset \mathbb{Z}^3$
  - probability of a configuration σ is

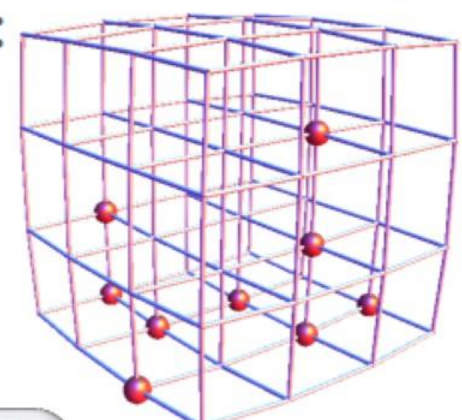
$$\mu(\sigma) = \frac{1}{Z_\beta} \exp\left(\beta \sum_{x\sim y} \sigma_x \sigma_y\right)$$



- The − cluster on bottom face:

SOS



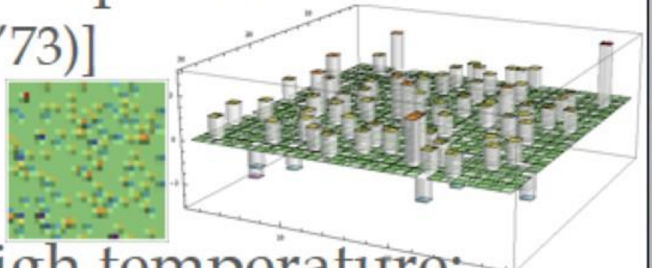


$$\pi_\Lambda(\eta) = \frac{1}{Z_\beta} \exp\left(-\beta \sum_{x\sim y} |\eta_x - \eta_y|\right)$$

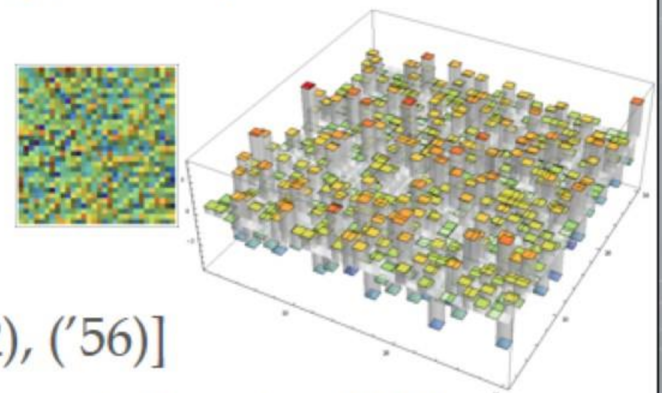
\# Ising disagreements
= ∑ height discrepancies

Roughening phase transition

- $\hat{\pi}_\Lambda$: SOS with heights in $\mathbb{Z}$ and boundary condition 0.
- 2D surface *localized* (rigid) at low temperature:
  - [Gallavotti, Martin-Löf, Miracle-Solé ('73)]
  - [Brandenberger, Wayne ('82)]

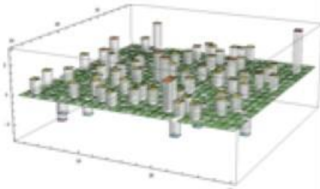


- 2D surface *delocalized* (rough) at high temperature:
  - [Fröhlich, Spencer ('81), ('83)].

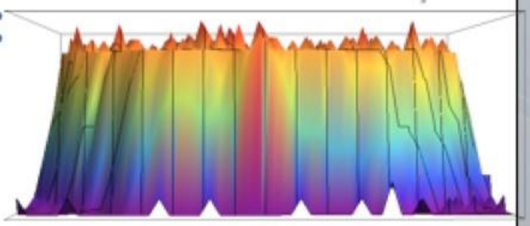


- Other dimensions:
  no roughening transition:
  - 1D: always rough [Temperley ('52), ('56)]
  - 3D: always rigid [Bricmont, Fontaine, Lebowitz ('82)].

Entropic repulsion



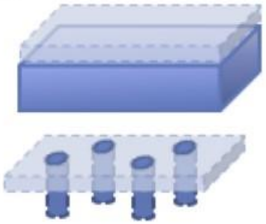
- At low temperature, the surface in $\hat{\pi}_\Lambda$ (*no wall*) is typically rigid. What is the effect of a **floor**?



- [Bricmont, El-Mellouki, Fröhlich (1986)]: Floor creates an *entropic repulsion* effect lifting the surface to height $\asymp \log L$.
  - "…the phenomenon of entropic repulsion: … the height of this surface above the wall diverges. This is an entropic effect. The surface has more freedom to fluctuate if it is far from the wall than if it is close to it."
  - Competing factors: the 0-boundary condition *vs.* the entropy from freedom to create *spikes downwards*.

Entropic repulsion (ctd.)

- Heuristic explanation [Bricmont *et al* ('86)]:
  - Suppose $\eta \equiv h - 1$. What is the cost of rising to h ?
    - ↑ Increase free energy by $4\beta L$ due to BC.
    - ↓ Decrease it by entropy term of $e^{-4\beta h} L^2$.
  - Equilibrium at $H \asymp \log L$.

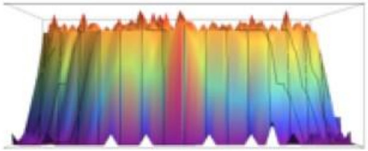


- THEOREM [Bricmont, El-Mellouki and Fröhlich (1986)]:

There $\exists\, c, C > 0$ such that for any large enough β,

$$(c/\beta) \log \Lambda \leq \frac{1}{\Lambda} \mathbb{E}_{\pi_\Lambda}[\textstyle\sum_x \eta_x] \leq (C/\beta) \log \Lambda .$$

  - Proof uses Pirogov-Sinai theory.



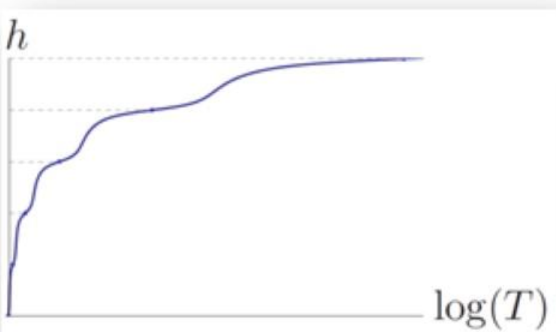
Recent progress

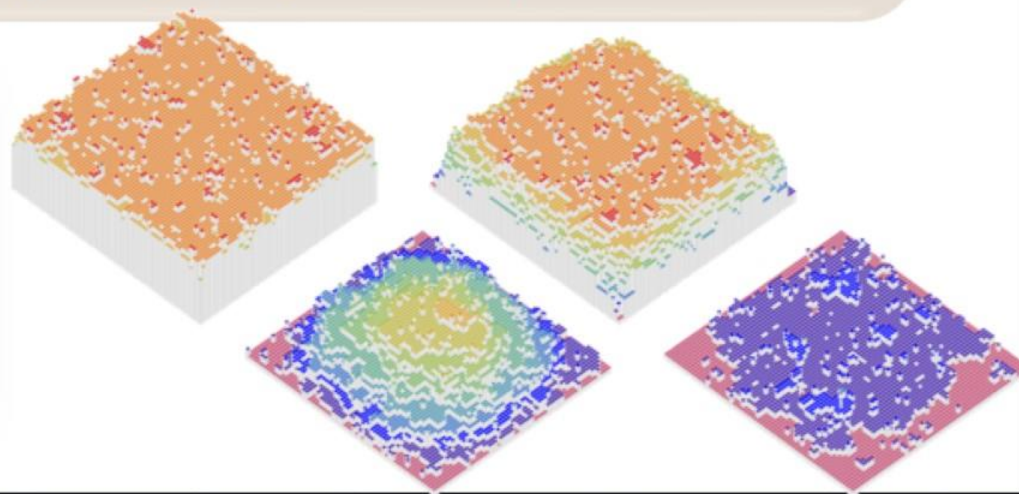
- [Caputo, L., Martinelli, Toninelli, Sly ’12]:

Most sites are at height $\frac{1}{4\beta}\log L + O(1)$.

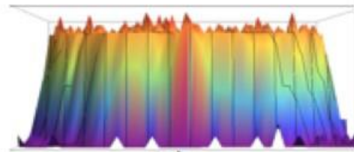
Dynamical Aspect

- Surface rises via a series of meta-stable states corresponding to formation of macroscopic droplets.
- Wait a time interval of $\exp\left(ce^{4\beta h}\right)$ to spawn a bubble of area $\frac{9}{10}L^2$ at level h , *e.g.*, $\exp(c\,L)$ near top level.

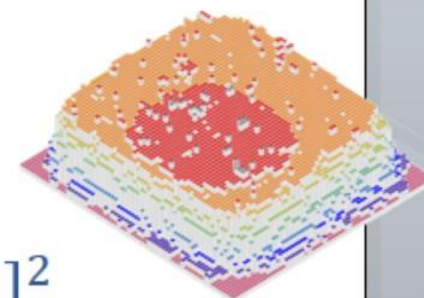




Basic question on (2+1)D SOS



- ***Height profile:***
  I. What is the average surface height? (diverges?)
  How concentrated are the heights?
  What is the maximum? (*doubled*?)



- ***Shape:*** rescale ensemble of level lines → $[0,1]^2$.
  II. Is there a scaling limit, *e.g.*, in Hausdorff distance?
  If so:
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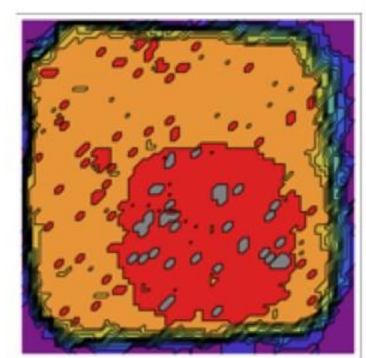
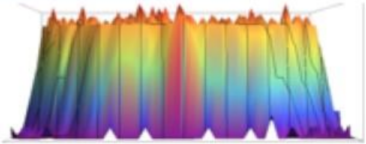
  IV. Quantitative estimates: for finite L what are the fluctuations of the level lines around their limit?



New understanding

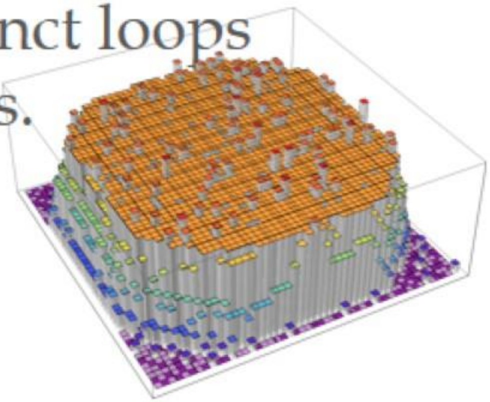
- [Caputo, L., Martinelli, Toninelli, Sly]:
  I. Height of most sites concentrates on a single level $H = \lfloor (4\beta)^{-1} \log L \rfloor$ for most values of L.
  (max is ~ $3/2$ the no-floor max…)



  II. For a sequence of diverging boxes:

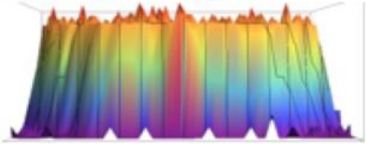
fractional parts of $(4\beta)^{-1} \log L$ converge to a noncritical value. $\Longrightarrow$ ensemble of level lines at heights $(H, H-1, \dots)$ has a scaling limit

  III. Limit explicitly given by nested distinct loops formed via translates of Wulff shapes.
  IV. Plateau has $L^{1/3+o(1)}$ fluctuations from wall centers.



Height of the surface

- Recall: [Bricmont, El-Mellouki, Fröhlich ('86)]:
  For β large: $(c/\beta)\log L \le \frac{1}{L^2}\mathbb{E}_{\pi_\Lambda}[\sum_x \eta_x] \le (C/\beta)\ \log L$.



- THEOREM [Caputo, L., Martinelli, Toninelli, Sly]:

Fix $\beta > 0$ large enough and set

$$H(L) = \lfloor (4\beta)^{-1}\log L \rfloor$$

$$E_h = \left\{\eta :\ \#\{x : \eta_x = h\} \ge \tfrac{9}{10}L^2\right\}$$

Then

$$\lim_{L\to\infty} \pi_\Lambda(E_{H-1} \cup E_H) = 1\ .$$

Plateau at one of two levels

- Recall: $H(L) = \lfloor (4\beta)^{-1} \log L \rfloor$
  $E_h = \{\eta : \#\{x : \eta_x = h\} \geq \frac{9}{10} L^2\}.$

  When does E_{H-1} occur as opposed to E_H ?

- THEOREM [Caputo, L., Martinelli, Toninelli, Sly]:

Fix $\beta > 0$ large enough and set

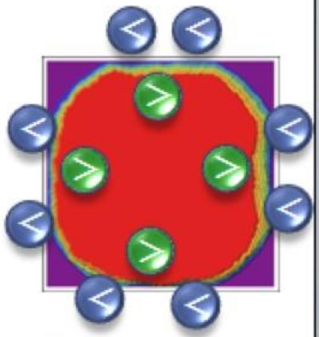
$$\alpha(L) = \mathrm{frac}\big((4\beta)^{-1} \log L\big)$$

$\leftarrow (4\beta)^{-1} \log L - H(L)$

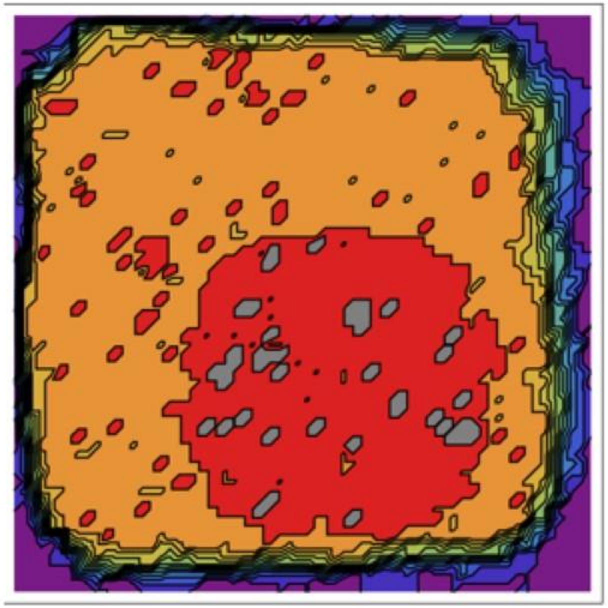
There exists $0 < \alpha_c < 1$ such that when $|\Lambda_n| \to \infty$:

1. If $\liminf_{n\to\infty} \alpha(L_n) > \alpha_c \Rightarrow \lim_{n\to\infty} \pi_{\Lambda_n}(E_H) = 1$
2. If $\limsup_{n\to\infty} \alpha(L_n) < \alpha_c \Rightarrow \lim_{n\to\infty} \pi_{\Lambda_n}(E_{H-1}) = 1$

The contour at the highest level



- DEFINITION: h-level line: collection of disjoint self-avoiding loops in the dual $\mathbb{Z}^{2*}$ formed by the set of bonds $\mathcal{E}_h = \{e' \in \mathbb{Z}^{2*}: \text{dual } e = (x,y) \text{ has } \eta_x \geq h,\ \eta_y < h\}$.



- Call a loop *microscopic* if its length is $\leq \log^2 L_n$ and *macroscopic* o/w.
- Thermal fluctuations give rise to microscopic loops. What can we say about the macroscopic ones?

Shape theorem

- Notation: $H_n(L_n) = \lfloor (4\beta)^{-1} \log L_n \rfloor$, $\alpha_n = \mathrm{frac}((4\beta)^{-1} \log L_n)$.
  $(\mathfrak{L}_0^n, \mathfrak{L}_1^n, \ldots)$ = loops at $(H_n, H_n - 1, \ldots)$ w. area $> \log^2 L_n$
  - Macroscopic *vs.* microscopic loops:

    W.h.p. above height H_n all loops are microscopic and every $\mathfrak{L}_i^n$ $(i \geq 1)$ has *one* loop.

  - Existence of scaling limit:

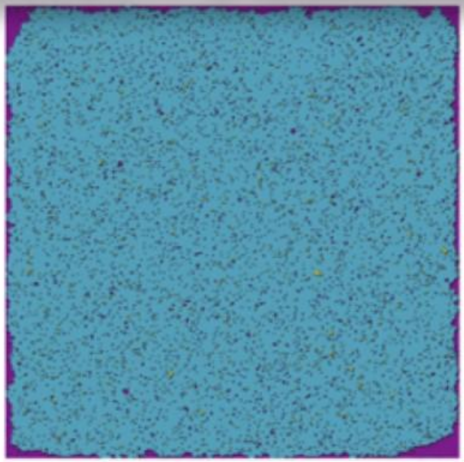
    $\frac{1}{L_n}(\mathfrak{L}_0^n, \mathfrak{L}_1^n, \ldots) \xrightarrow{a.s.} (\mathcal{W}_0, \mathcal{W}_1, \ldots)$ if $\exists \lim_{n\to\infty} \alpha_n \neq \alpha_c$.
    (Precisely: $\limsup_{n\to\infty} \sup_i d_{\mathcal{H}}\left(\frac{1}{L_n}\mathfrak{L}_i^n, \mathcal{W}_i\right) = 0$ a.s.)

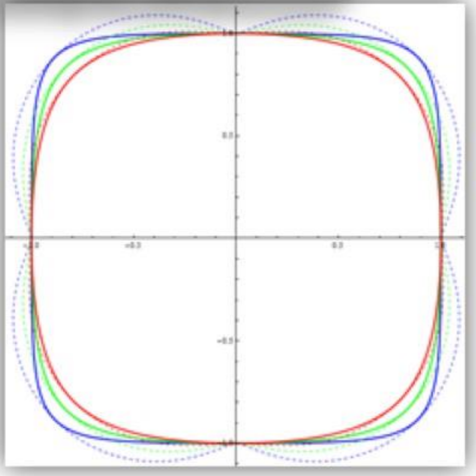
Shape theorem (ctd.)

- Description of the limit:

Nested distinct loops invariant under $\pi/4$ rotations:
$\mathcal{W}_i = \bigcup\{\text{all translates of } \mathcal{W}_\star(r_i)\}$ for explicit $r_i \searrow 0$,
$\mathcal{W}_\star(x) = x$-dilation of a smooth convex shape $\mathcal{W}_\star$.

$\mathcal{W}_\star$





Wulff shape for $\beta = \frac{3}{2}, 2, 3$

- *E.g.*, the plateau has:
  - Distance of $\asymp L$ from corners.
  - Distance of $o(L)$ from center sides. *More precisely*?

Cube-root fluctuations

- Notation:

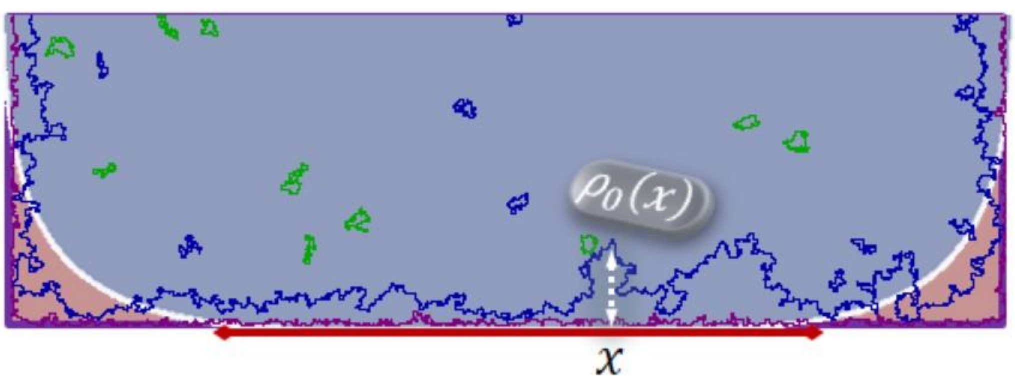
$\mathfrak{L}_0$: macroscopic loop at height $H(L) = \lfloor (4\beta)^{-1} \log L \rfloor$

$\rho_0(x)$: vertical distance of $\mathfrak{L}_0$ from bottom at coordinate x

I_0 : intersection of its scaling limit & bottom boundary

- THEOREM: For any $\varepsilon > 0$, w.h.p.

$$L^{1/3-\varepsilon} < \sup_{x \in I_0} \rho_0(x) < L^{1/3+\varepsilon}$$

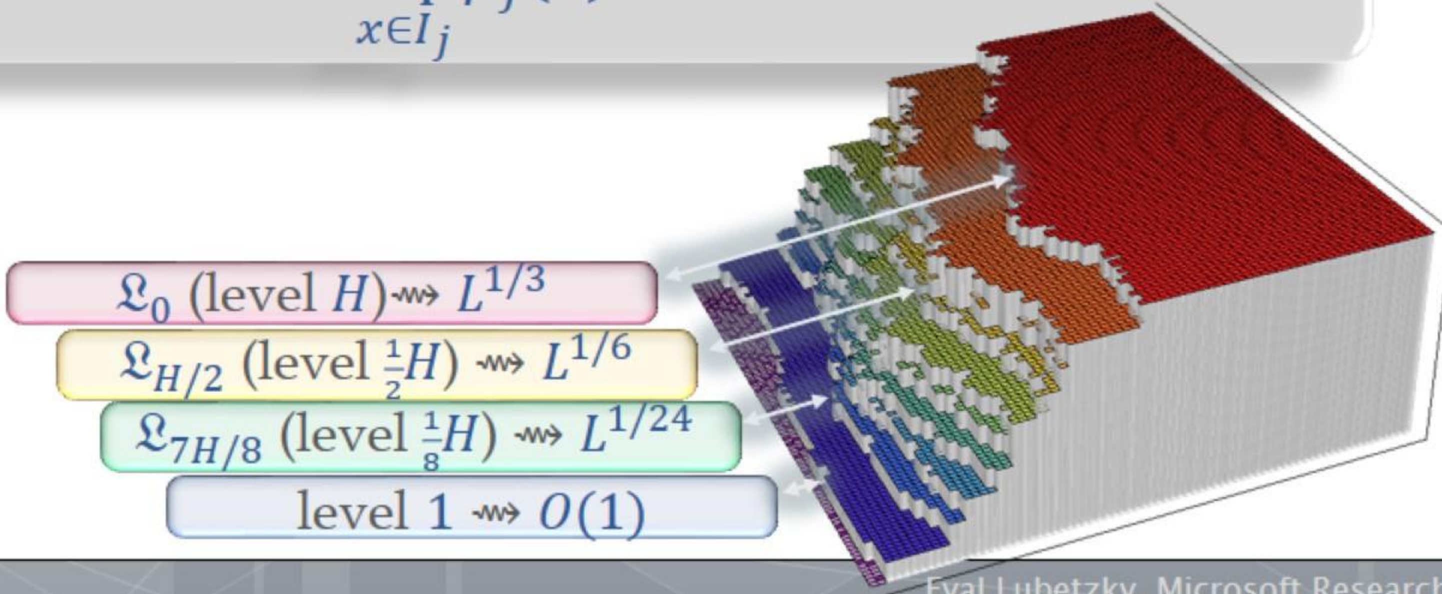


Cascade of fluctuation exponents

- Interpolating the side-boundaries fluctuations:
  - top level : $L^{1/3+o(1)}$, … , bottom level : $O_{\mathrm{P}}(1)$
- DEF: $\rho_j(x)$ = distance of $\mathfrak{L}_j$ from bottom at coordinate x
- COROLLARY:

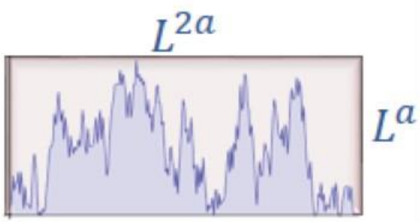
Let $0 < \xi < 1$ and $j = \lfloor \xi H \rfloor$. For any $\varepsilon > 0$, w.h.p.

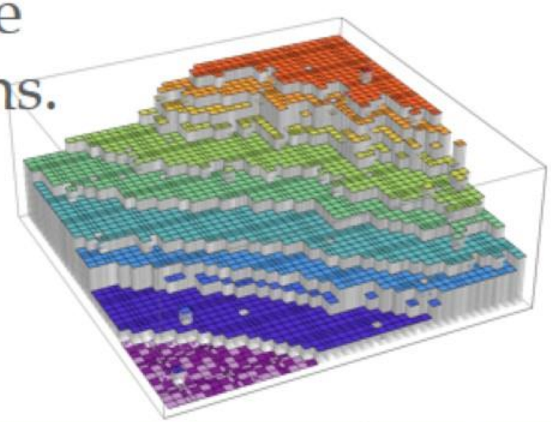
$$\sup_{x \in I_j} \rho_j(x) < L^{(1-\xi)/3+\varepsilon}$$



Cube–root fluctuations

- Reason behind the $L^{1/3}$ fluctuations:
  - h-contour behaves like a RW reweighted by an area term of $\exp\left(c\, e^{-4\beta h} \text{Area}(\gamma)\right)$.
  - Take a $L^{2a} \times L^{a}$ rectangle at level $H = (4\beta)^{-1} \log L$:
    - Area term = $\exp\left(-\frac{c'}{L} \text{Area}(\gamma)\right)$.
    - For $3a < 1$: Area term negligible $\Rightarrow$ SRW behavior of L^a fluctuations.





Some ideas from the proofs

- Key estimate (simplified): for any γ and $h > 0$:

$$\pi_\Lambda(\gamma \text{ is an } h\text{-contour}) \le \exp\left(-\beta|\gamma| + ce^{-4\beta h}\,\mathrm{Area}(\gamma)\right)$$

- Simpler case: no floor:

<u>LEMMA</u>:

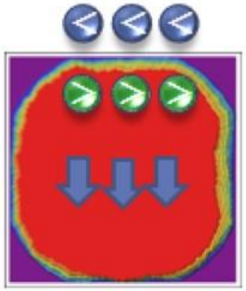
$$\hat\pi_\Lambda(\gamma \text{ is an } h\text{-contour}) \le \exp(-\beta|\gamma|) \qquad \forall \gamma, h > 0$$

<u>PROOF</u>:

- Let T be the map on the space of configurations Ω :

$$(T\eta)(x) = \begin{cases} \eta_x - 1 & x \in \Lambda_\gamma \\ \eta_x & x \notin \Lambda_\gamma \end{cases}$$

- If γ is an h-contour then $\hat\pi_\Lambda(T\eta) = e^{\beta|\gamma|}\hat\pi_\Lambda(\eta)$
- $\Rightarrow$ event has probability at most $e^{-\beta|\gamma|}$.



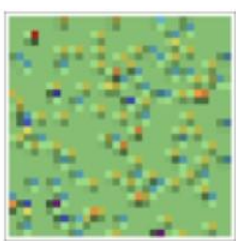
∎

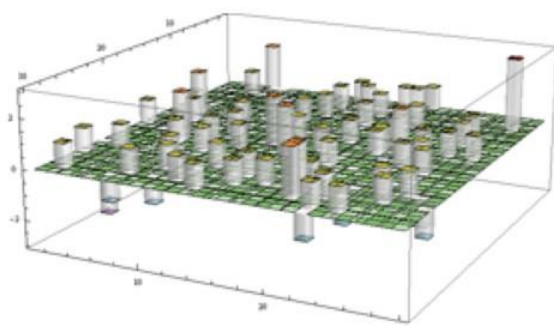
Some ideas from the proofs (ctd.)

- Key estimate (simplified): for any γ and $h > 0$:

$$\pi_\Lambda(\gamma \text{ is an } h\text{-contour}) \le \exp\left(-\beta|\gamma| + ce^{-4\beta h}\,\text{Area}(\gamma)\right)$$

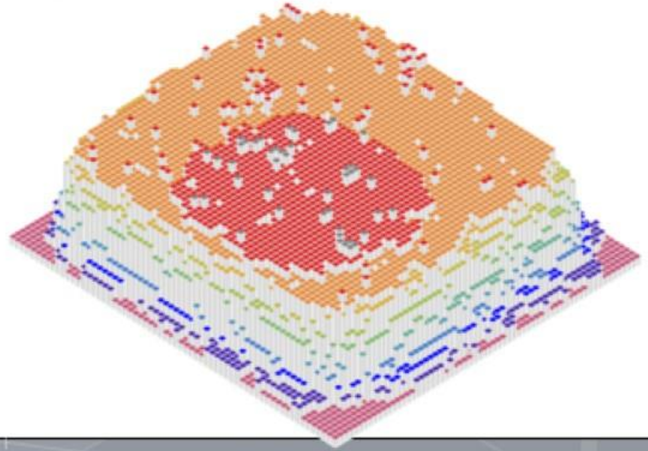
- In the presence of a floor:
  - The map T is no longer injective.
  - Argument still valid provided that $\eta > 0$, *i.e.*, *no sites are at level 0*.
  - This brings the **area** term into play! (*one direction: FKG; other direction: …*)





Summary and open problems

- SOS approximation of low-temperature 3D Ising:
  - Evolves through a series of metastable states.
  - Plateau at height H or $H-1$.
  - Ensemble of level lines converges to a sequence of nested loops formed by translates of Wulff shapes.
  - Plateau has $L^{1/3+o(1)}$ fluctuations from side-boundaries.
  - Cascade of exponents in intermediate level lines
  - …
- ? Critical behavior?
- ? Lower bound on fluctuations of intermediate levels?



Thank you