Markov Chain Minicourse

Summer school in Probability

lecture 4

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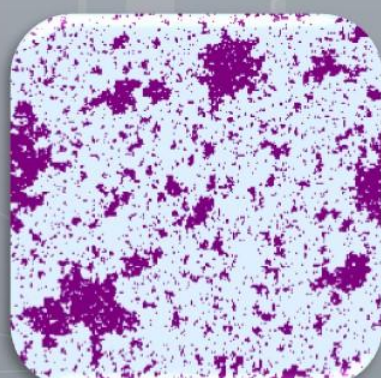
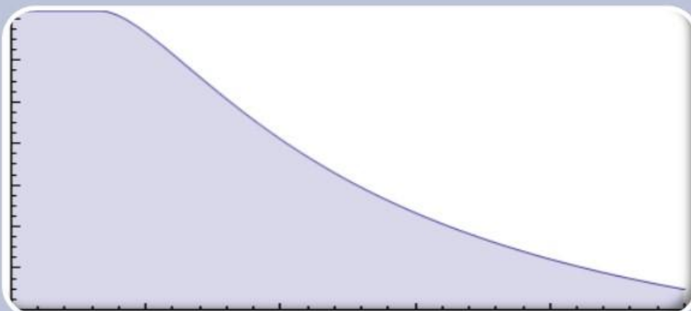
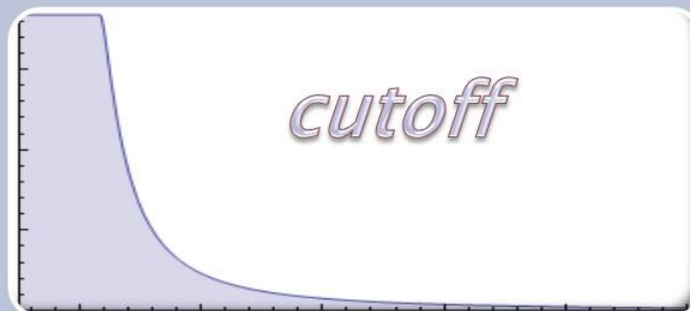


The Cutoff Phenomenon

- Describes a sharp transition in the convergence of finite ergodic Markov chains to stationarity.





Steady convergence
it takes a while to reach distance ½ from stationarity then a while longer to reach distance ¼, etc.

Abrupt convergence
distance from equilibrium quickly drops from 1 to 0

Cutoff: formal definition

- A family of chains (X_t^n) is said to have ***cutoff*** if:

$$\lim_{n\to\infty} \frac{t_{\mathrm{mix}}(\varepsilon)}{t_{\mathrm{mix}}(1-\varepsilon)} = 1 \quad \forall\ 0 < \varepsilon < 1.$$

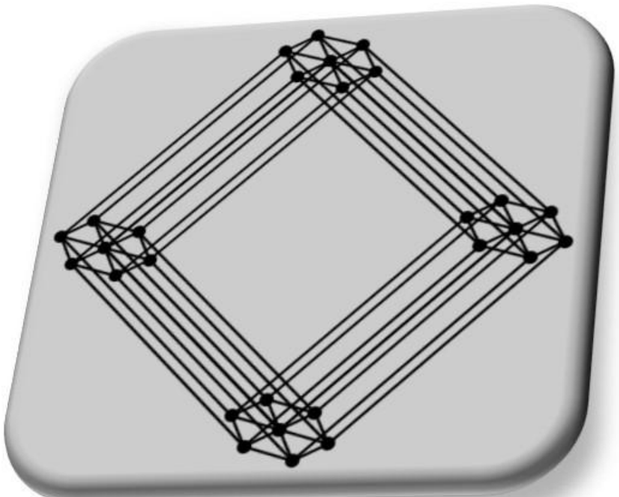
i.e., $t_{\mathrm{mix}}(\alpha) = (1+o(1))\,t_{\mathrm{mix}}(\beta)$ for any $0 < \alpha\,,\beta < 1$.

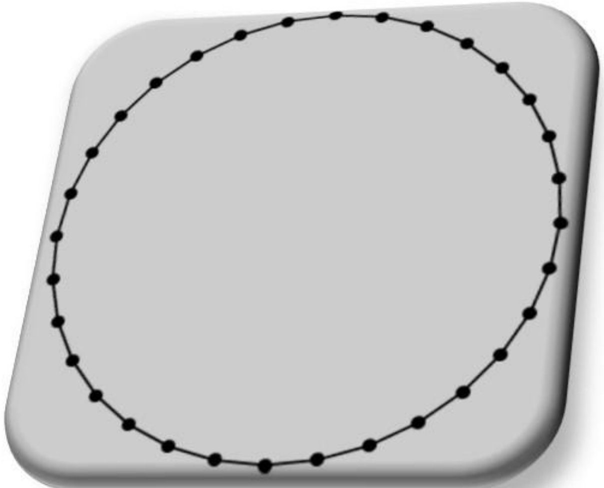
- A sequence (w_n) is called a ***cutoff window*** if

$$w_n = o(t_{\mathrm{mix}}(\tfrac{1}{4}))\ ,$$
$$t_{\mathrm{mix}}(\varepsilon) - t_{\mathrm{mix}}(1-\varepsilon) = O_\varepsilon(w_n) \quad \forall\, 0 < \varepsilon < 1\ .$$

Basic examples

Lazy discrete-time simple random walk



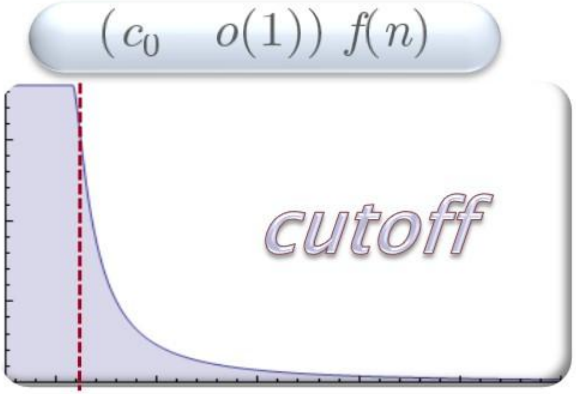


On the hypercube $\{0,1\}^n$:
☑ Exhibits cutoff at
$\frac{1}{2}\, n \log n + O(n)$
[Aldous '83]

On the n–cycle:
☒ No cutoff.

The importance of cutoff

- Suppose we run Glauber dynamics for the Ising Model satisfying $t_{\text{mix}} \asymp f(n)$ for some $f(n)$.
- Cutoff $\Leftrightarrow$ $\exists$ some $c_0 > 0$ so that:
  - Must run the chain for at least $\sim c_0 \cdot f(n)$ steps to even reach distance $(1-\varepsilon)$ from μ.
  - Running it any longer than that is essentially redundant.



- Proofs usually require (and thus provide) a deep understanding of the chain (its reasons for mixing).
- Many natural chains are *believed* to have cutoff, yet proving cutoff can be extremely challenging.

Cutoff History

- Random walks on graphs and groups:
  - Discovered:
    - Random transpositions on S_n [Diaconis, Shahshahani '81]
    - RW on the hypercube, Riffle-shuffle [Aldous '83]
  - Named "Cutoff Phenomenon" in the top-in-at-random shuffle analysis [Diaconis, Aldous '86]
  - RWs on finite groups [Saloff-Coste '04]
  - RWs on random regular graphs [L., Sly '10]
- One-dimensional Markov chains:
  - Birth-and-Death chains [Diaconis, Saloff-Coste '06], [Ding, L., Peres '09]
- No proofs of cutoff except when stationary distribution is completely understood and has many symmetries [*till recently*]

Peres' Product Criterion

- QUESTION [Diaconis '96]: How can we determine whether a given Markov chain exhibits cutoff?
- OBSERVATION [Peres '04]: if a reversible chain has cutoff then $\mathrm{gap} \cdot t_{\mathrm{mix}}(1/4) \to \infty$
  or equivalently: $t_{\mathrm{rel}} = o\big(t_{\mathrm{mix}}(1/4)\big)$.
- PROOF:
  - Key fact: every reversible Markov chain satisfies
    $$t_{\mathrm{mix}}(\varepsilon) \geq (t_{\mathrm{rel}} - 1)\log(\tfrac{1}{2\varepsilon}).$$
  - Assume that $t_{\mathrm{rel}} \geq 1+\delta\, t_{\mathrm{mix}}(1/4)$ for some $\delta > 0$.
  - It follows that $t_{\mathrm{mix}}(\varepsilon) \geq f(\varepsilon) \cdot t_{\mathrm{mix}}(1/4)$ where $f(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \infty$
    $\Rightarrow$ No (pre) cutoff. ∎

Peres' Product Criterion (ctd.)

- The condition ⊗ is necessary for cutoff.
  Is it also sufficient, giving a method to determine the existence of cutoff?
- [Aldous '04]: unfortunately *not*: the product-condition ⊗ does not imply cutoff (explicit construction).
- Even so, Peres conjectured that for many natural families of chains, *cutoff* occurs iff ⊗ .
  (*e.g.*, holds for birth-and-death chains [Ding, L., Peres '09]).

⊗ $\mathrm{gap} \cdot t_{\mathrm{mix}}\left(\frac{1}{4}\right) \to \infty$ ⇐ *cutoff*, ⇒ ?

- Notable conjectured examples:
  - Ising on lattices ; Potts model on lattices;
  - Gas Hard-core model on lattices; lattice Colorings ;
  - Anti-ferromagnetic Ising / Potts model, Spin-glass,
  - Arbitrary boundary conditions / external field; ...

Recently: cutoff for Ising on lattices

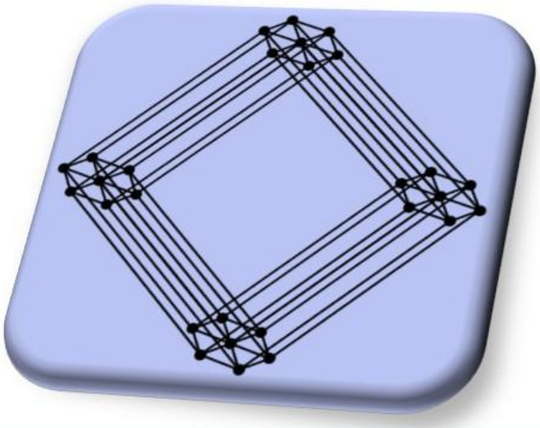
- <u>THEOREM</u> [L., Sly]:

Let $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$ be the critical inverse-temperature for the Ising model on $\mathbb{Z}^2$. Then the continuous-time Glauber dynamics for the Ising model on $(\mathbb{Z}/n\mathbb{Z})^2$ with periodic boundary conditions at $0 \leq \beta < \beta_c$ has cutoff at $(1/\lambda_\infty)\log n$ where λ_∞ is the spectral gap of the dynamics on the infinite volume lattice.

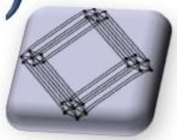
- Analogous result holds for *any* dimension $d \geq 1$:
  - Cutoff at $(d/2\lambda_\infty)\log n$.
  - *E.g.*, cutoff at $[2(1-\tanh(2\beta))]^{-1}\log n$ for $d=1$.

Random walk on the hypercube

- Glauber dynamics for infinite temperature (β=0) Ising $\equiv$ lazy RW on the hypercube $\{$-1,1$\}^n$:
  - Stationary distribution is uniform.
  - Spins evolve independently.
- [Aldous '83]: Cutoff at $\frac{1}{2}n\log n + O(n)$.
  - Twice faster than trivial upper bound.
  - Constant window in continuous time version.



Cutoff for RW on hypercube (ctd.)



- Symmetry $\Rightarrow$ Start at the all-plus state.
- Symmetry $\Rightarrow$ Mixing of *magnetization* $S_t = \sum_{i=1}^{n} X_t(i)$ [a birth & death chain] determines entire mixing:

$$\left\|\mathbb{P}_+(X_t \in \cdot) - \pi\right\|_{\mathrm{TV}} = \left\|\mathbb{P}_+(S_t \in \cdot) - \pi_S\right\|_{\mathrm{TV}} .$$

  - To bound the coupling-time of this 1d chain it thus suffices to couple it from its extreme ends $+$, $-$.
- Magnetizations contract to within $\sqrt{n}$ from each other:

$$\mathbb{E}_+[S_t] = n(1-\tfrac{1}{n})^t \;,\; \mathbb{E}_-[S_t] = -n(1-\tfrac{1}{n})^t .$$

  - At time $t = \frac{1}{2} n \log n$ the expected distance between the chains is $O(\sqrt{n})$.
- Afterwards: distance is a biased RW drifting towards 0. Comparing to SRW $\Rightarrow$ takes $O(n)$ further steps to hit 0.

$\blacksquare$