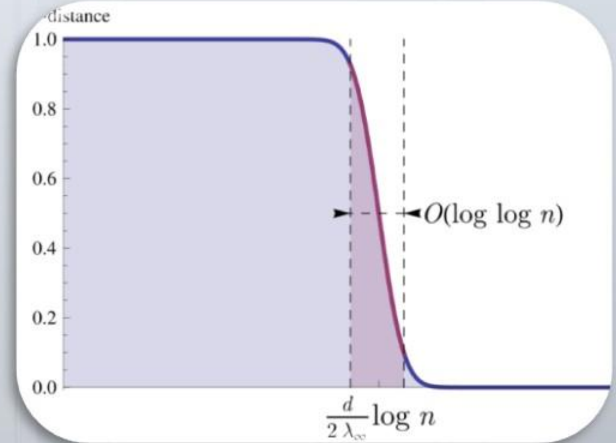

La Pietra 2011
Mini course
lecture 5

Cutoff for Ising on the lattice

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Microsoft Research

Upper bound via sparse sets

- We showed:

$$\forall s \geq \frac{10d}{\alpha_s} \log\log n \;\; \forall\, W_s: \quad \begin{array}{l} \mathbb{P}\left(\Delta_{W_s} \in \mathcal{S}\right) \geq 1 - O\left(n^{-10d}\right) \\ \mathbb{P}\left(u \in \Delta_{W_s}\right) \leq \log^{-5d} n \quad \forall u \end{array}$$

$\forall s \leq \log^{43} n \;\forall\, t \;\forall\, \sigma_0$:

$$\left\| \mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{\mathrm{TV}} \leq \int \left\| \mathbb{P}_{\sigma_0}(X_t(\Delta_{W_s}) \in \cdot) - \mu|_{\Delta_{W_s}} \right\|_{\mathrm{TV}} d\mathbb{P}(W_s) + O(n^{-10d})$$

- <u>COROLLARY</u>:

Let $t > 0$ and $\frac{10d}{\hat{\alpha}_s} \log\log n \leq s \leq \log^{4/3} n$. Then $\exists$ measure ν on the sparse sets $\mathcal{S}$ s.t. $\nu(\{\Delta : u \in \Delta\}) < \log^{-5d} n \;\; \forall u$ and

$$\left\| \mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{\mathrm{TV}} \leq \int_{\mathcal{S}} \left\| \mathbb{P}_{\sigma_0}(X_t(\Delta) \in \cdot) - \mu|_{\Delta} \right\|_{\mathrm{TV}} d\nu(\Delta) + O(n^{-10d})$$

The projection onto a sparse set

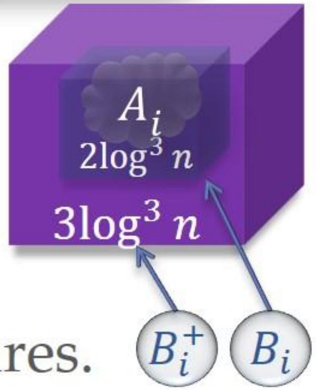
- LEMMA:

Let $\Delta \in \mathcal{S}$ be a *sparse* set and $A_1, \dots, A_{N_\Delta}$ be its component partition. Then for $\forall \sigma_0$ and $t \le t_0$,

$$\left\| \mathbb{P}_{\sigma_0}(X_t(\Delta) \in \cdot) - \mu|_\Delta \right\|_{\mathrm{TV}} \le \left\| \mathbb{P}_{\sigma_0}(\bar{X}_t^*(\cup B_i) \in \cdot) - \mu^*|_{\cup B_i} \right\|_{\mathrm{TV}} + O(n^{-10d})$$

where $(\bar{X}_t^*)$ is the product chain on N_Δ i.i.d. cubes B_i^+

- PROOF:
  - Couple $X_t(\Delta)$ to $\bar{X}_t^*(\Delta)$ via $A_i^+ = B_{A_i}(\log^{3/2} n)$ to agree throughout $t \in [0, \log^{4/3} n]$.
  - Inspect $\bar{X}_t^*(\Delta)$ started from equilibrium at time $t_0 = \log^{4/3} n$ to couple stationary measures.
  - Decrease projection from Δ to $\cup B_i$ to conclude proof. ∎



Concluding the upper bound

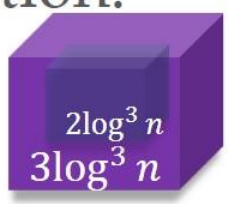
- So far we showed:

Let $t \le \log^{4/3} n$ and $\frac{10d}{\hat{\alpha}_s} \log\log n \le s \le \log^{4/3} n$. Then $\exists$ measure ν on $\mathcal{S}$ s.t. $\nu(\{\Delta : u \in \Delta\}) < \log^{-5d} n \quad \forall u$ and

$$\left\| \mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{\mathrm{TV}} \le \int \left\| \mathbb{P}_{\sigma_0}(\bar{X}_t^*(\cup B_i) \in \cdot) - \mu^*|_{\cup B_i} \right\|_{\mathrm{TV}} d\nu(\Delta) + O(n^{-10d})$$

- For $\Delta \in \mathcal{S}$ with N_Δ comp. apply Product Proposition:

$$\max_{\sigma_0} \left\| \mathbb{P}_{\sigma_0}(\bar{X}_t^*(\cup B_i) \in \cdot) - \mu^*|_{\cup B_i} \right\|_{\mathrm{TV}} \le \sqrt{\mathfrak{M}_t}$$

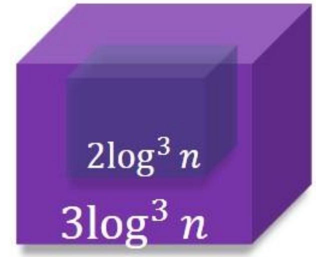


where $\mathfrak{M}_t = N_\Delta \mathfrak{m}_t$ and $\mathfrak{m}_t = \left\| \mathbb{P}_{\sigma_0}(X_t^*(B) \in \cdot) - \mu^*|_B \right\|^2_{L^2(\mu^*|_B)}$

- Integrate to get:

$$\max_{\sigma_0} \left\| \mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{\mathrm{TV}} \le \left((n/\log^5 n)^d \, \mathfrak{m}_t \right)^{1/2} + O(n^{-10d})$$

Existence of cutoff



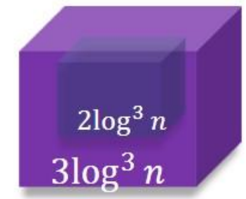
- Framework:
  - (X_t) : Glauber dynamics for $\mathbb{Z}_n^d$
  - (X_t^*) : Glauber dynamics on $\mathbb{Z}_r^d$ for $r = 3\log^3 n$.
  - B : smaller cube within $\mathbb{Z}_r^d$ of side-length $2\log^3 n$.
- For $t \leq \log^{4/3} n$ and $10d\hat{\alpha}_s^{-1}\log\log n \leq s \leq \log^{4/3} n$:

$$\max_{\sigma_0}\left\|\mathbb{P}_{\sigma_0}(X_{t+s}\in\cdot)-\mu\right\|_{\mathrm{TV}} \leq \left((n/\log^5 n)^d\,\mathfrak{m}_t\right)^{1/2} + O(n^{-10d})$$

- Matching lower bound: take order $(n/\log^3 n)^d$ such cubes (well-spaced) to get that for $t > 20d\hat{\alpha}_s^{-1}\log\log n$:

$$\max_{\sigma_0}\left\|\mathbb{P}_{\sigma_0}(X_t\in\cdot)-\mu\right\|_{\mathrm{TV}} \geq f\left((n/\log^3 n)^d\,\mathfrak{m}_t\right) - O(n^{-10d}) \quad \text{where} \quad \lim_{x\to\infty} f(x) = 1$$

Existence of cutoff



- Recall: $\mathfrak{m}_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0}(X^*_t(B) \in \cdot) - \mu^*|_B \right\|^2_{L^2(\mu^*|_B)}$

  and choose: $\begin{cases} t^\star \triangleq \inf\left\{t : \mathfrak{m}_t \le n^{-d} \log^{3d+1} n\right\}, \\ s \triangleq 10d\, \hat{\alpha}_s^{-1} \log\log n \ . \end{cases}$

  B exp. mixed in L^2 at $t^\star$

  (by log-Sobolev inequalities $t^\star \asymp \log\ n$).

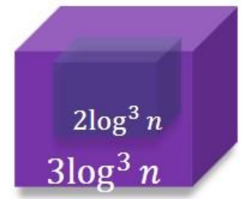
- By def.: $\begin{cases} (n/\log^3 n)^d\, \mathfrak{m}_{t^\star} = \log n \to \infty \\ (n/\log^5 n)^d\, \mathfrak{m}_{t^\star} = \log^{1-2d} n \to 0 \end{cases}$

- Conclude: $\max_{\sigma_0} \left\| \mathbb{P}_{\sigma_0}(X_{t^\star+s} \in \cdot) - \mu \right\|_{\mathrm{TV}} \le \log^{\frac{1}{2}-d} n + O(n^{-10d}) \to 0$

  $\max_{\sigma_0} \left\| \mathbb{P}_{\sigma_0}(X_{t^\star} \in \cdot) - \mu \right\|_{\mathrm{TV}} \ge f(\log n) - O(n^{-10d}) \to 1$

- Entire mixing occurs at interval $(t^\star, t^\star + s)$, i.e. cutoff at time $t^\star$ with window $\le s = O(\log\ \log\ n)$. ∎

Establishing cutoff location



- Recall:

$$\mathfrak{m}_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0}\left(X_t^*(B) \in \cdot\right) - \mu^*|_B \right\|^2_{L^2(\mu^*|_B)}$$

$$t^\star \triangleq \inf\left\{t : \mathfrak{m}_t \leq n^{-d} \log^{3d+1} n\right\}.$$

- <u>LEMMA</u>:

Set $c_0 = \frac{12d}{\hat{\alpha}_s}$, $\frac{10d}{\hat{\alpha}_s \hat{\lambda}} \log\log n \leq t \leq \log^{4/3} n$ and $r = 3\log^3 n$.

Then: $e^{-\lambda(r)t - c_0 \log\log n} - O(n^{-10d}) \leq \mathfrak{m}_t \leq e^{-\lambda(r)t + c_0 \log\log n}$

- <u>COROLLARY</u>: cutoff in terms of local spectral gap:

$$t^\star = \frac{d}{2\lambda(r)} \log n \quad \text{and window} \quad \leq \frac{40d}{\hat{\alpha}_s \hat{\lambda}} \log\log n$$

With some extra work

- The limit as $n \to \infty$ of $\lambda(n)$, spectral gap of Ising on $\mathbb{Z}_n^d$ exists and equals λ_∞, the gap of infinite-vol. dynamics.
  - Convergence rate obtain from proof:
  $$|\lambda(n) - \lambda_\infty| < n^{-1/2+o(1)}$$
- Other boundary conditions e.g. all-plus:
  - Analyze locations of cubes in the system how many are adjacent to the outer boundary
  - Obtain an expression in terms of mixed eigenvalues, e.g. for $\mathbb{Z}_n^2$ with ⊕ b.c. ∃ cutoff at
  $$t^\star = (\lambda_\infty \wedge 2\lambda_{\mathbb{H}})^{-1} \log n$$

■

General (believed) picture for the Glauber dynamics

- Setting: Ising model on the lattice $(\mathbb{Z}/n\mathbb{Z})^d$.
  Belief: For some critical inverse-temperature β_c :
- Low temperature: ($\beta > \beta_c$)
  gap^{-1} and t_{mix} are ***exponential*** in the surface area.
- Critical temperature: ($\beta = \beta_c$)
  gap^{-1} and t_{mix} are ***polynomial*** in the surface area.
  - Exponent of gap^{-1} is universal (the *dynamical critical exponent* z).
- High temperature: ($\beta < \beta_c$)
  - ***Rapid*** mixing: $\text{gap}^{-1} = O(1)$ and $t_{\text{mix}} \asymp \log n$
  - Mixing occurs abruptly, *i.e.* there is ***cutoff*** .

Mixing on the square lattice

- High & low temperature regimes fully settled:

$\beta < \beta_c$	β_c	$\beta > \beta_c$
$\text{gap}^{-1} = O(1)$ $t_{\text{mix}} = O(\log n)$		$\text{gap}^{-1} = \exp(c(\beta)+o(1))n$ $t_{\text{mix}} = \exp(c(\beta)+o(1))n$

- Power law at the critical $\beta = \beta_c = \frac{1}{2}\log(1+\sqrt{2})$?
  - Numerical experiments: universal exponent of ~ 2.17
    [Ito '93], [Wang, Hatano, Suzuki '95], [Grassberger '95], [Nightingale, Blöte '96], [Wang, Hu '97],…
  - **No previous sub-exponential upper bounds** …
  - Only geometries with analogous critical power-law:
    Mean-field [Ding, L., Peres '09], Regular tree [Ding, L., Peres '10]

Recent result

- **THEOREM:** ([L.-Sly]

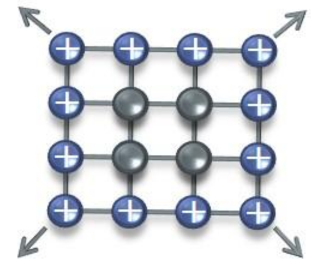
Consider Glauber dynamics for the critical Ising model on a finite $n \times n$ box $\Lambda \subset \mathbb{Z}^2$ with an arbitrary boundary condition τ. There exists an absolute $C > 0$ (independent of Λ, τ) so that the mixing of the dynamics is at most n^C.

- **COROLLARY:**

Perfect simulation (zero error approximation) for the 2D critical Ising model with arbitrary boundary conditions.

Best previous samplers gave *approximated* sample for *specific* homogenous boundaries [JS'93], [RW'96].

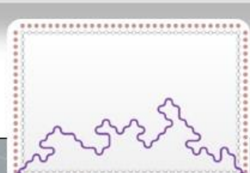
Another recent result

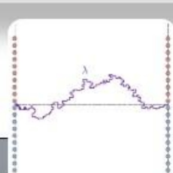


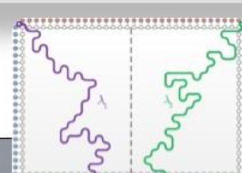
- Mixing believed to be polynomial…
  - [Martinelli '94]: mixing $\leq \exp(n^{1/2+o(1)})$ for $n \times n$ box in $\mathbb{Z}^2$ and large enough β.
  - Breakthrough by [Martinelli, Toninelli '10]: $t_{\text{mix}} \leq \exp(n^{\epsilon})$ for any $\epsilon > 0$ and large enough β. *(Sub-exponential)*
- More recently: *Quasi-polynomial down to β_c*

  <u>**Theorem**</u>: ([L., Martinelli, Toninelli, Sly]:

Consider Glauber dynamics for the Ising model on an $n \times n$ box $\Lambda \subset \mathbb{Z}^2$ with all-plus boundary. For any $\beta > \beta_c$ there exists $C(\beta) > 0$ so that the mixing time is at most $n^{C \log n}$.







Glauber dynamics on $\mathbb{Z}^2$

- Present state-of-the-art bounds on mixing:

$\beta < \beta_c$

$$\mathrm{gap}^{-1} = O(1)$$
$$t_{\mathrm{mix}} = \tfrac{1}{2}\lambda_\infty^{-1} \log n + O(\log\log n)$$

Cutoff

β_c

$$n^{7/4} \le \mathrm{gap}^{-1} \le t_{\mathrm{mix}} \le n^c$$

$\beta > \beta_c$

Free *b.c. :*

$$\mathrm{gap}^{-1} = \exp(c(\beta) + o(1))n$$
$$t_{\mathrm{mix}} = \exp(c(\beta) + o(1))n$$

Plus *b.c. :*

$$\mathrm{gap}^{-1} \le t_{\mathrm{mix}} \le n^{O(\log n)}$$

Open problems

- Ising model on the 2D lattice…
  - Calculate the precise *dynamical critical exponent*.
  - Show the Glauber dynamics is *polynomial* at low temperatures under all-plus b.c.

- Ising model on the 3D lattice…
  - Establish strong spatial mixing throughout the high temperature regime (zero external field).
  - Establish power-law behavior at criticality and sub-exponential low-temp mixing under all-plus b.c.

Thank you