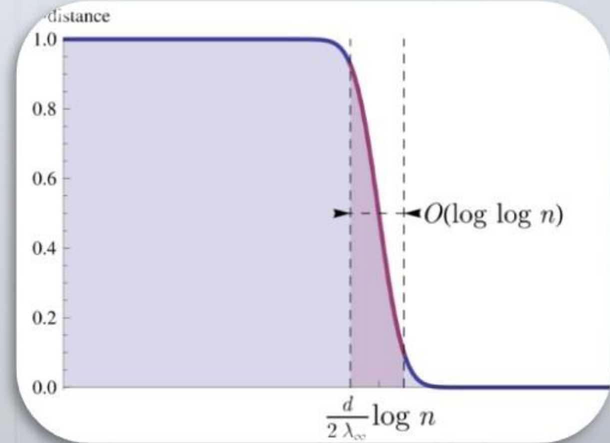
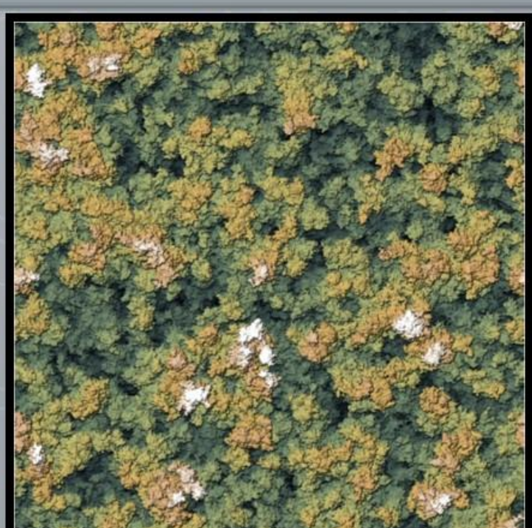


La Pietra 2011
Mini course
lectures 3,4

Cutoff for Ising on the lattice



Eyal Lubetzky
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Recap: product chains $L^1 \to L^2$ reduction

- PROPOSITION:

Let $X_t = (X_t^1, \dots, X_t^n)$ be a product chain where each X_t^i is ergodic with stationary measures π_i and $\pi = \prod_i \pi_i$. Let

$$\mathfrak{M}_t = \sum_{i=1}^{n} \mathfrak{m}_t \quad \text{where} \quad \mathfrak{m}_t = \left\| \mathbb{P}(X_t^i \in \cdot) - \pi_i \right\|_{L^2(\pi_i)}^2 .$$

For $\forall\, \delta > 0$ there $\exists\, \varepsilon > 0$ so that if for some $t > 0$

$$\max_i \left\| \mathbb{P}(X_t^i \in \cdot) - \pi_i \right\|_{L^\infty(\pi_i)} < \varepsilon$$

then

$$\left| \left\| \mathbb{P}(X_t \in \cdot) - \pi \right\|_{\mathrm{TV}} - \left(2\Phi\left(\tfrac{1}{2}\sqrt{\mathfrak{M}_t}\right) - 1 \right) \right| < \delta .$$

Products of i.i.d.'s

- COROLLARY:

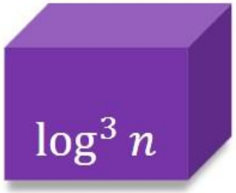
Let X_t be a product chain made of n i.i.d. copies of a finite ergodic chain Y_t with spectral-gap and log-Sobolev const gap and α_s resp. and stationary measure φ . If

$$\log \varphi_{\min}^{-1} \leq n^{o(\alpha_s/\mathrm{gap})}$$

then X_t exhibits cutoff at $\frac{1}{2}\,\mathrm{gap}^{-1} \log n$ with window of order $O(\alpha_s^{-1} \log_+ \log \varphi_{\min}^{-1})$.

Intuition: cutoff on the lattice

- Break up $\mathbb{Z}_n^d$ to cubes of side-length $\log^3 n$. Dynamics on such a cube:
  - $\alpha_s^{-1} = O(1)$
  - $\log \varphi_{\min}^{-1}(\sigma) = O(\log^{3d} n) = n^{o(1)}$



- Take non-adjacent cubes $Q_1, \ldots, Q_m$ ($m \asymp (n/\log^3 n)^d$) and ***suppose as if*** the projection on those would predict mixing for the entire system:
  - Distance between cubes turn them $\approx$ independent.
  - Expect cutoff at $\frac{1}{2\mathrm{gap}} \log m = \frac{1}{2\mathrm{gap}} \log n + O(\log\log n)$ with window $O(\log\log n)$.

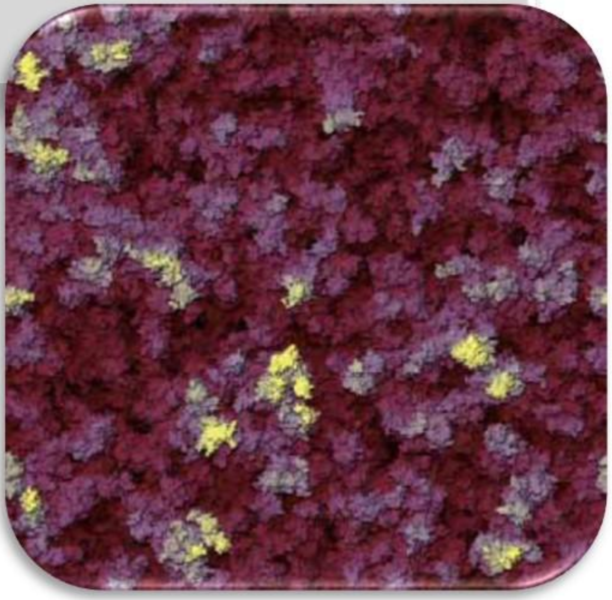
Making this rigorous: sparse sets

- DEFINITION:

The set $\Lambda \subset V$ is ***sparse*** iff it can be partitioned into (not necessarily connected) components $\{A_i\}$ so that

(i) $\mathrm{diam}(A_i) = O(\log^3 n)$ (ii) $\mathrm{dist}(A_i, A_j) \geq \log^2 n$

Let $\mathcal{S} = \{\Lambda \subset V \;:\; \Lambda \text{ is sparse}\}$.



- Motivation:
  - Small diameter $\rightsquigarrow$ can embed each component in a small box.
  - Super logarithmic distances between components $\rightsquigarrow$ essentially independent.

Upper bound via sparse sets

- <u>THEOREM</u>:

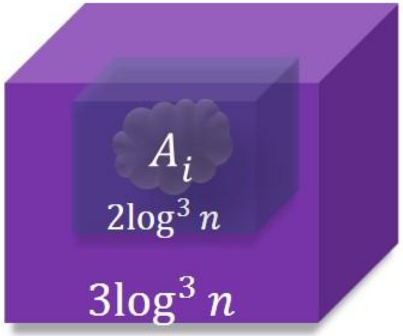
Let $t > 0$ and $\frac{10d}{\hat{\alpha}_s} \log\log n \le s \le \log^{4/3} n$. Then $\exists$ measure ν on the sparse sets $\mathcal{S}$ s.t. $\nu(\{\Delta : u \in \Delta\}) < \log^{-5d} n \quad \forall u$ and

$$\left\| \mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{\mathrm{TV}} \le \int_{\mathcal{S}} \left\| \mathbb{P}_{\sigma_0}(X_t(\Delta) \in \cdot) - \mu|_{\Delta} \right\|_{\mathrm{TV}} d\nu(\Delta) + O(n^{-10d})$$

- *Assuming theorem*, from here we can:
  - Box each component A_i (extended a bit) inside B_i then extend to a larger box.
  - Couple dynamics to a product chain agreeing on the projections on $\cup B_i$

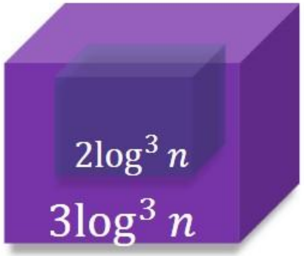
Identical

Independent



L^1–L^2 reduction for Ising

- Framework:
  - (X_t) : Glauber dynamics for $\mathbb{Z}_n^d$
  - (X_t^*) : Glauber dynamics on $\mathbb{Z}_r^d$ for $r = 3\log^3 n$.
  - B : smaller cube within $\mathbb{Z}_r^d$ of side-length $2\log^3 n$.



- Define:

$$\mathfrak{m}_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0}(X_t^*(B) \in \cdot) - \mu^*|_B \right\|^2_{L^2(\mu^*|_B)}$$

(measure L^2 convergence of the projection $(X_t^*) \hookrightarrow B$.)

- There are $m \asymp (n/\log^3 n)^d$ such disjoint cubes in $\mathbb{Z}_n^d$, so as a lower bound take the proposition with

$$\mathfrak{M}_t \triangleq (n/\log^3 n)^d \, \mathfrak{m}_t$$

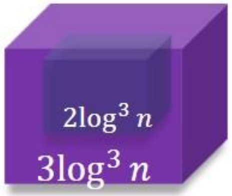
L^1–L^2 reduction for Ising (ctd.)

- Recall: $$\mathfrak{m}_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0}(X_t^*(B) \in \cdot) - \mu^*|_B \right\|^2_{L^2(\mu^*|_B)}$$

- <u>THEOREM</u>:

Suppose $$\begin{cases} 10d\,\hat{\alpha}_s^{-1} \log\log n \le s < \log^{4/3} n \\ 20d\,\hat{\alpha}_s^{-1} \log\log n \le t < \log^{4/3} n \end{cases}$$
where $\hat{\alpha}_s$ is the infimum over log-Sobolev constants.
Then
$$(n/\log^5 n)^d \mathfrak{m}_t \to 0 \;\Rightarrow\; \limsup_{n\to\infty} \max_{x_0} \left\| \mathbb{P}_{x_0}(X_{t+s} \in \cdot) - \mu \right\|_{\mathrm{TV}} = 0$$
$$(n/\log^3 n)^d \mathfrak{m}_t \to \infty \Rightarrow \liminf_{n\to\infty} \max_{x_0} \left\| \mathbb{P}_{x_0}(X_t \in \cdot) - \mu \right\|_{\mathrm{TV}} = 1$$

Existence of cutoff



- Recall: $\mathfrak{m}_t \triangleq \max_{x_0} \left\| \mathbb{P}_{x_0}(X_t^*(B) \in \cdot) - \mu^*|_B \right\|^2_{L^2(\mu^*|_B)}$

  and choose: $\begin{cases} t^\star \triangleq \inf\left\{t : \mathfrak{m}_t \leq n^{-d} \log^{3d+1} n\right\}, \\ s \triangleq 10d\,\hat{\alpha}_s^{-1} \log\log n \,. \end{cases}$

- By def.: $\begin{cases} (n/\log^3 n)^d \,\mathfrak{m}_{t^\star} = \log n \to \infty \\ (n/\log^5 n)^d \,\mathfrak{m}_{t^\star} = \log^{1-2d} n \to 0 \end{cases}$

- Remains to check range of $t^\star$:
  - Due to log-Sobolev inequalities $t^\star \asymp \log\, n$

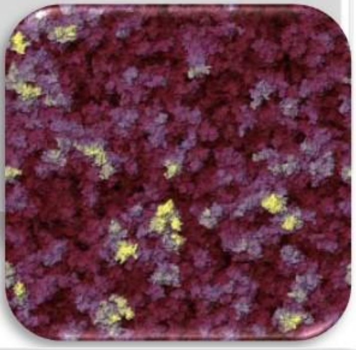
- By Theorem: entire mixing occurs at interval $(t^\star, t^\star + s)$
  $\Rightarrow$ cutoff at time $t^\star$ with window $\leq s$.

Sparse sets upper bound

- DEFINITION:

The set $\Lambda \subset V$ is ***sparse*** ($\Lambda \in \mathcal{S}$) if it can be partitioned into (not necessarily connected) components $\{A_i\}$ so that

1. $\mathrm{diam}(A_i) \leq \frac{1}{2}\log^3 n$
2. $\mathrm{dist}(A_i, A_j) \geq \log^2 n$



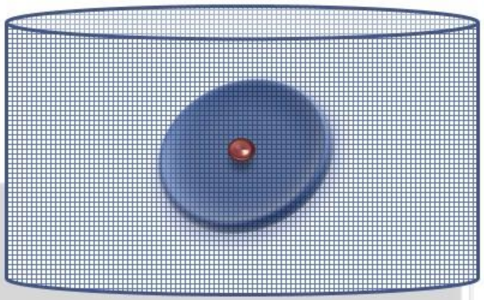
- THEOREM:

Let $t > 0$ and $\frac{10d}{\hat{\alpha}_s}\log\log n \leq s \leq \log^{4/3} n$. Then $\exists$ measure ν on the sparse sets $\mathcal{S}$ s.t. $\nu(\{\Delta: u \in \Delta\}) < \log^{-5d} n \quad \forall u$ and

$$\left\|\mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu\right\|_{\mathrm{TV}} \leq \int_{\mathcal{S}} \left\|\mathbb{P}_{\sigma_0}(X_t(\Delta) \in \cdot) - \mu|_\Delta\right\|_{\mathrm{TV}} d\nu(\Delta) + O(n^{-10d})$$

Barrier dynamics

- Random map $\mathcal{G}_s\colon \Omega \to \Omega$ (where $\Omega = \{\pm 1\}^V$) coupled to the Glauber dynamics.

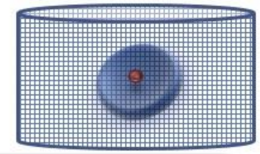


- <u>DEFINITION</u>

For $s > 0$ define $\mathcal{G}_s(X_0)$ as follows:

- Surround $\forall\, u \in V$ by $B_u(\log^{3/2} n)$, a ball of radius $\log^{3/2} n$ by graph metric.
- Impose periodic boundary (“barrier”) on each ball.
- Run standard dynamics (X_t) till time s and use same site-choices and unit-variables for updates.
- Output: the spins at centers of $\{B_u(\log^{3/2} n) : u \in V\}$

Working with the barrier dynamics



- LEMMA:

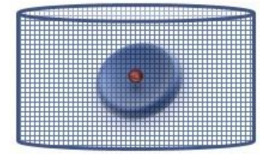
> The barrier dynamics map $\mathcal{G}_s$ can be coupled to the original Glauber dynamics X_t such that
> $$\mathbb{P}\left(X_s = \mathcal{G}_s(X_0) \;\; \forall s \in [0, \log^{4/3} n]\right) \geq 1 - n^{-10d}.$$

- PROOF:
  - Use implicit coupling defining the barrier dynamics.
  - Disagreement at $u \Rightarrow$ sequence of updates at times $t_1 < \cdots < t_\ell < \log^{4/3} n$ connects $u \leftrightsquigarrow \partial B_u(\log^{3/2} n)$:

$$\mathbb{P}\Big(\bigcup_{u,t}\{X_t(u) \neq \tilde{X}_t(u)\}\Big) \leq n^d \sum_{\ell \geq \log^{3/2} n} (2d)^\ell \, \mathbb{P}(\mathrm{Po}(\log^{4/3} n) \geq \ell)$$

$$\leq C n^d e^{-c \log^{3/2} n} < n^{-10d}. \quad \blacksquare$$

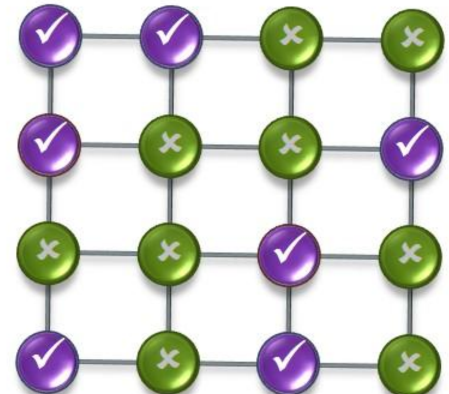
Update support



- Update sequence for the barrier dynamics map $\mathcal{G}_s$ in interval $[0, s]$:
  - Seq. of triplets (t_i, x_i, u_i)

    time *site* *unit var*

  - Given this: $\mathcal{G}_s = g_{W_s}$ det. monotone.



- DEFINITION:

Let W_s = update seq. for barrier dynamics map $\mathcal{G}_s$.
The ***support*** of W_s is the minimum subset $\Delta_{W_s} \subset V$ s.t. $g_{W_s}(\sigma_0)$ is determined by $\sigma_0(\Delta_{W_s})$ for $\forall \sigma_0$.

- Equiv.: $x \in \Delta_{W_s}$ if $\exists \sigma_0$ such that $g_{W_s}(\sigma_0) \neq g_{W_s}(\sigma_0^x)$.

Upper bound via update support

- LEMMA:

Let W_s = random update seq. of the barrier dynamics map in the interval $(0,s)$ for some $s \le \log^{4/3} n$. Then $\forall \sigma_0 \ \forall t > 0$

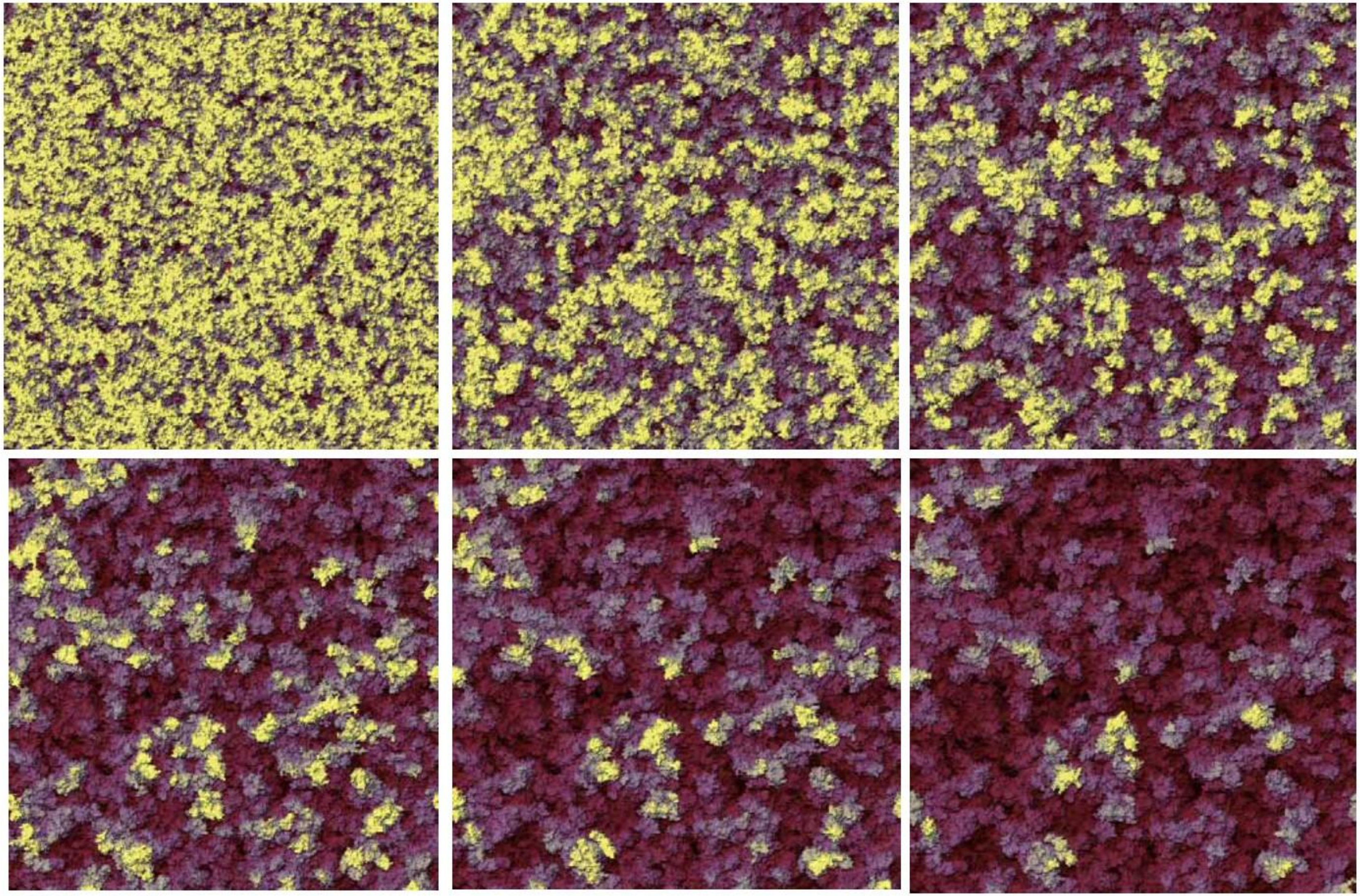
$$\left\| \mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{\mathrm{TV}} \le \int \left\| \mathbb{P}_{\sigma_0}(X_t(\Delta_{W_s}) \in \cdot) - \mu|_{\Delta_{W_s}} \right\|_{\mathrm{TV}} d\mathbb{P}(W_s) \ + O(n^{-10d})$$

- PROOF:
  - Couple dynamics to two instances of the barrier dynamics run for time s.
  - Reduce to an integral over L^1 distances between the deterministic barrier-dynamics functions.
  - Projection can only decrease L^1 distance. ■

Update support is sparse

- Most supports are sparse:
  - *Volume decays exponentially*
  - *Components separated and small*
- As time traverses, the effect of more and more sites becomes 0 (information flow stops at barriers of barrier dynamics).

Random support of update seq.



Update support is sparse (ctd.)

- LEMMA:

Let W_s be the random update sequence of the barrier dynamics in the interval $(0, s)$ for some $s \geq \frac{10d}{\alpha_s} \log\log n$.

Then $\mathbb{P}(\Delta_{W_s} \in \mathcal{S}) \geq 1 - O(n^{-10d})$

and $\mathbb{P}(u \in \Delta_{W_s}) \leq \log^{-5d} n \ \ \forall u$.

- PROOF:
  - Estimate the probability that a full copy $B_u(\log^{3/2} n)$ of the barrier-dynamics is "trivial" (coupling).
  - No long ($\varepsilon \log n$) path of nontrivial balls by a first moment argument.

∎

Upper bound via sparse sets

- We showed:

$$\forall s \geq \frac{10d}{\alpha_s} \log\log n \ \forall\, W_s : \begin{array}{l} \mathbb{P}\left(\Delta_{W_s} \in \mathcal{S}\right) \geq 1 - O\left(n^{-10d}\right) \\ \mathbb{P}\left(u \in \Delta_{W_s}\right) \leq \log^{-5d} n \quad \forall u \end{array}$$

$\forall s \leq \log^{43} n \ \forall\, t \ \forall\, \sigma_0$:

$$\left\| \mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{\mathrm{TV}} \leq \int \left\| \mathbb{P}_{\sigma_0}(X_t(\Delta_{W_s}) \in \cdot) - \mu|_{\Delta_{W_s}} \right\|_{\mathrm{TV}} d\mathbb{P}(W_s) + O(n^{-10d})$$

- <u>COROLLARY</u>:

Let $t > 0$ and $\frac{10d}{\hat{\alpha}_s} \log\log n \leq s \leq \log^{4/3} n$. Then $\exists$ measure ν on the sparse sets $\mathcal{S}$ s.t. $\nu(\{\Delta : u \in \Delta\}) < \log^{-5d} n \ \ \forall u$ and

$$\left\| \mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{\mathrm{TV}} \leq \int_{\mathcal{S}} \left\| \mathbb{P}_{\sigma_0}(X_t(\Delta) \in \cdot) - \mu|_{\Delta} \right\|_{\mathrm{TV}} d\nu(\Delta) + O(n^{-10d})$$

The projection onto a sparse set

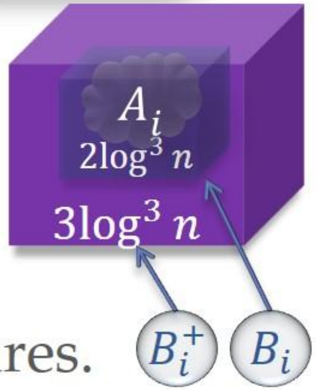
- LEMMA:

Let $\Delta \in \mathcal{S}$ be a *sparse* set and $A_1, \dots, A_{N_\Delta}$ be its component partition. Then for $\forall \sigma_0$ and $t \le t_0$,

$$\left\| \mathbb{P}_{\sigma_0}(X_t(\Delta) \in \cdot) - \mu|_\Delta \right\|_{\mathrm{TV}} \le \left\| \mathbb{P}_{\sigma_0}(\bar{X}_t^*(\cup B_i) \in \cdot) - \mu^*|_{\cup B_i} \right\|_{\mathrm{TV}} + O(n^{-10d})$$

where $(\bar{X}_t^*)$ is the product chain on N_Δ i.i.d. cubes B_i^+

- PROOF:
  - Couple $X_t(\Delta)$ to $\bar{X}_t^*(\Delta)$ via $A_i^+ = B_{A_i}(\log^{3/2} n)$ to agree throughout $t \in [0, \log^{4/3} n]$.
  - Inspect $\bar{X}_t^*(\Delta)$ started from equilibrium at time $t_0 = \log^{4/3} n$ to couple stationary measures.
  - Decrease projection from Δ to $\cup B_i$ to conclude proof. ∎



Concluding the upper bound

- So far we showed:

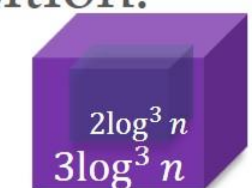
Let $t \le \log^{4/3} n$ and $\frac{10d}{\hat{\alpha}_s} \log\log n \le s \le \log^{4/3} n$. Then $\exists$ measure ν on $\mathcal{S}$ s.t. $\nu(\{\Delta : u \in \Delta\}) < \log^{-5d} n \ \ \forall u$ and

$$\left\| \mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{\mathrm{TV}} \le \int \left\| \mathbb{P}_{\sigma_0}(\bar{X}_t^*(\textstyle\bigcup B_i) \in \cdot) - \mu^*|_{\bigcup B_i} \right\|_{\mathrm{TV}} d\nu(\Delta) + O(n^{-10d})$$

- For $\Delta \in \mathcal{S}$ with N_Δ comp. apply Product Proposition:

$$\max_{\sigma_0} \left\| \mathbb{P}_{\sigma_0}(\bar{X}_t^*(\textstyle\bigcup B_i) \in \cdot) - \mu^*|_{\bigcup B_i} \right\|_{\mathrm{TV}} \le \sqrt{\mathfrak{M}_t}$$

where $\mathfrak{M}_t = N_\Delta \mathfrak{m}_t$ and $\mathfrak{m}_t = \left\| \mathbb{P}_{\sigma_0}(X_t^*(B) \in \cdot) - \mu^*|_B \right\|^2_{L^2(\mu^*|_B)}$



- Integrate to get:

$$\max_{\sigma_0} \left\| \mathbb{P}_{\sigma_0}(X_{t+s} \in \cdot) - \mu \right\|_{\mathrm{TV}} \le \left((n/\log^5 n)^d \, \mathfrak{m}_t \right)^{1/2} + O(n^{-10d})$$