Critical slowdown for the Ising model on the 2D lattice

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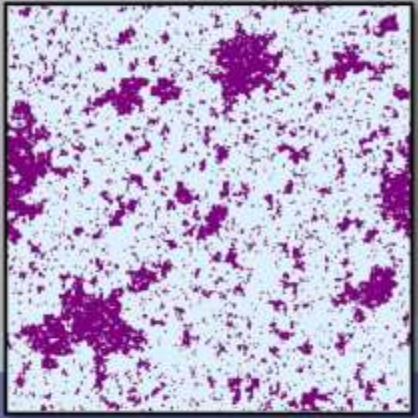
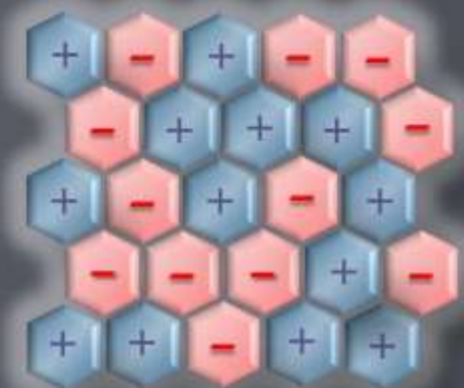


Joint work w. Allan Sly

Ising model

- Underlying geometry: finite graph $G=(V,E)$.
- Set of possible configurations:

$$\Omega = \{\pm 1\}^V$$



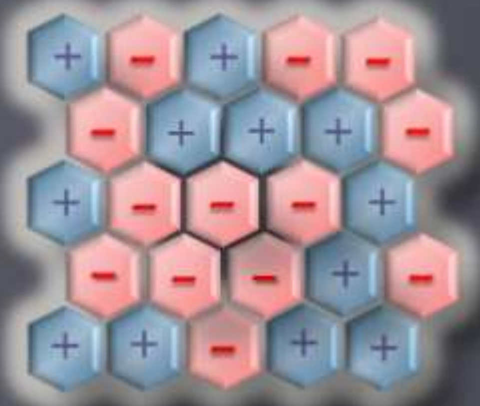
- Probability of a configuration $\sigma \in \Omega$ given by the *Gibbs distribution*

$$\mu(\sigma) = \frac{1}{Z(\beta)} \exp\left(\beta \sum_{xy \in E} \sigma(x)\sigma(y)\right)$$ [no external field]

- *Ferromagnetic* $\leftrightsquigarrow$ inverse-temperature $\beta \geq 0$.
- Phase transition as β varies (in some geometries).

Glauber dynamics for Ising

- One of the most commonly used MC samplers for the Gibbs distribution:
  - Update sites via *iid* Poisson(1) clocks
  - Each update replaces a spin at $u \in V$ by a new one $\sim \mu$ conditioned on $V \setminus \{u\}$ (heat-bath version).



- Ergodic reversible MC with stationary measure μ.
- Introduced by Glauber in 1963. Other versions of the dynamics include e.g. Metropolis.
- *How fast does it converge to equilibrium?*

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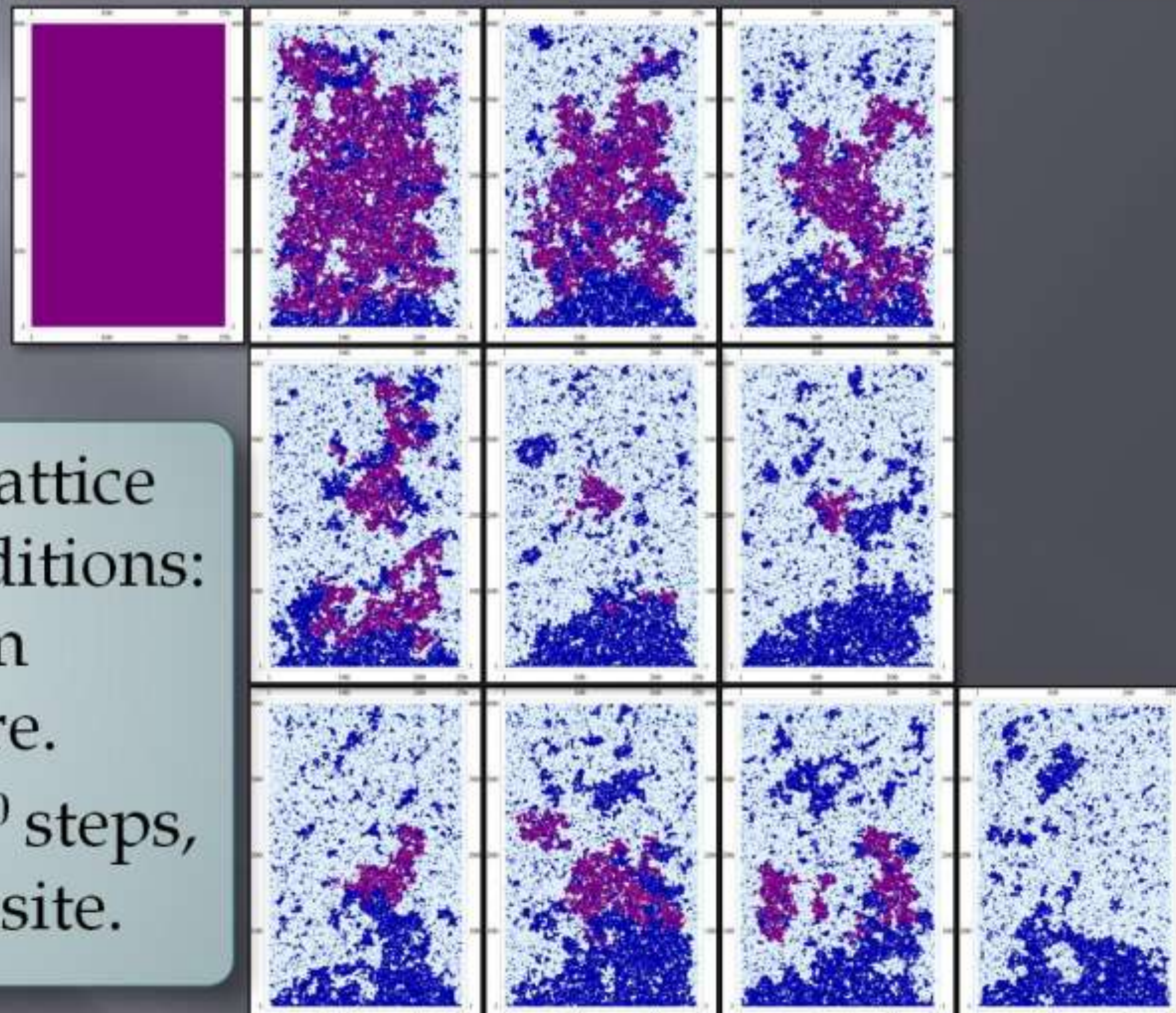
Example: Glauber dynamics for critical Ising on the square lattice



- 256 x 400 square lattice w. boundary conditions:
  (+) at bottom
  (−) elsewhere.
- Frame every $\sim 2^{30}$ steps, i.e. $\sim 2^{13}$ updates/site.

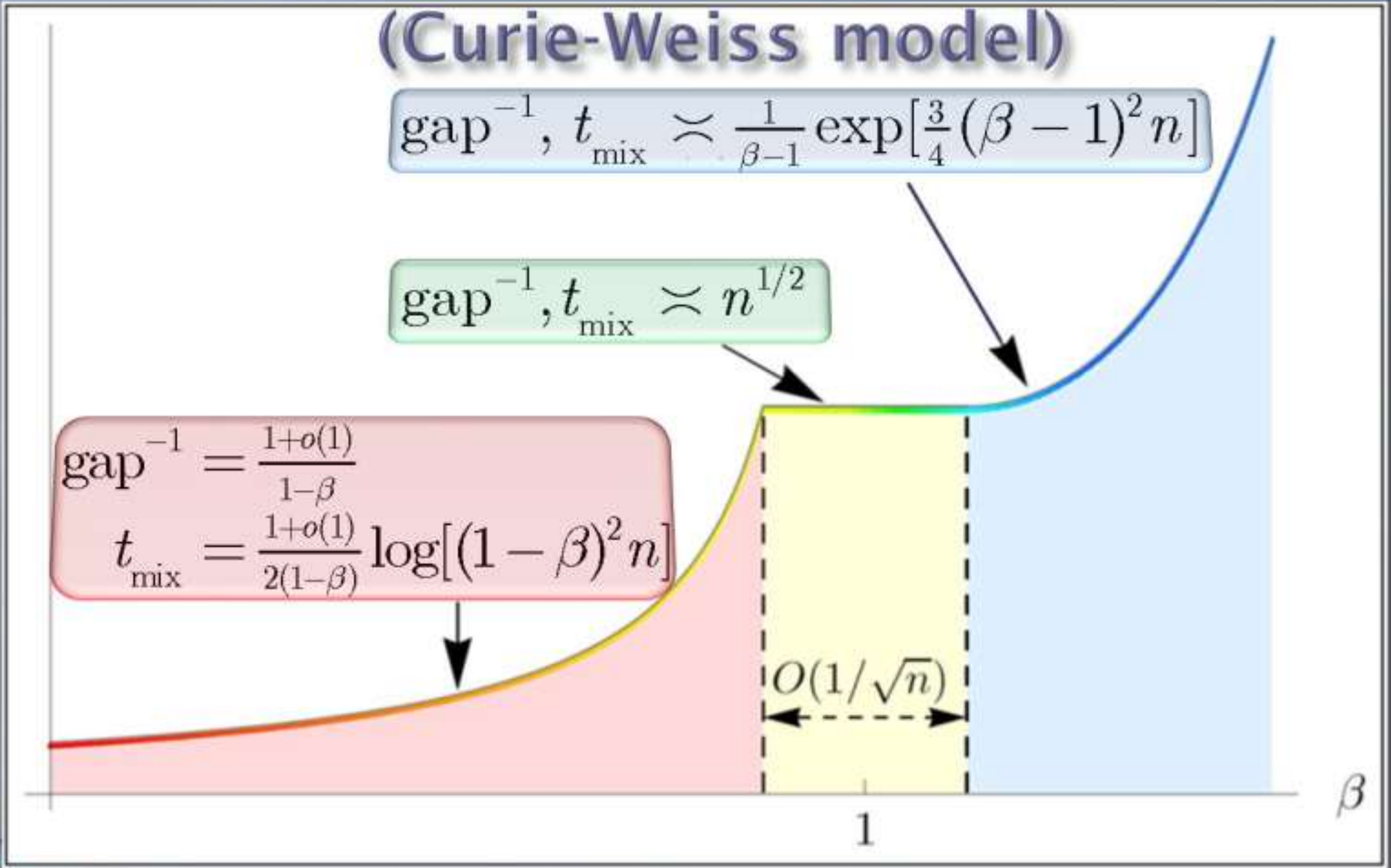
Rate of convergence to equilibrium

- Spectral gap in the spectrum of the generator:
  gap = smallest positive eigenvalue of the heat-kernel H_t of the dynamics.
  - Governs convergence in $L^2(\mu)$.
  - Dirichlet-form characterization: $\text{gap} = \inf_f \frac{\mathcal{E}(f)}{\text{Var}(f)}$
    where
    $$\mathcal{E}(f) = \left\langle \mathcal{L}f, f \right\rangle_{L^2(\mu)} = \frac{1}{2}\sum_{\sigma,x} \mu(\sigma) c(x,\sigma)\left[f(\sigma^x) - f(\sigma)\right]^2 .$$
- Mixing time : standard measure of convergence:
  - The L^1 (total-variation) mixing time within ε is
    $$t_{\text{mix}}(\varepsilon) = \inf\left\{t : \max_\sigma \left\| H_t(\sigma,\cdot) - \mu\right\|_{\text{TV}} \le \varepsilon\right\} .$$

General (believed) picture for Glauber dynamics

- Setting: Ising model on the lattice $(\mathbb{Z}/n\mathbb{Z})^d$.
  Belief: For some critical inverse-temperature β_c :
- Low temperature: $(\beta > \beta_c)$
  gap^{-1} and t_{mix} are ***exponential*** in the surface area.
- Critical temperature: $(\beta = \beta_c)$
  gap^{-1} and t_{mix} are ***polynomial*** in the surface area.
- High temperature: $(\beta < \beta_c)$
  1. ***Rapid*** mixing: $\text{gap}^{-1} = O(1)$ and $t_{\text{mix}} \asymp \log n$
  2. Mixing occurs abruptly (***cutoff*** phenomenon).

Gap/mixing-time evolution for Ising on mean-field



Above picture established in [Ding, L., Peres '09].

Mixing time for Ising on lattices: High temperature regime

- Mixing time of Ising on the lattice at high temp. was established in a series of seminal papers:
  - [Aizenman, Holley '84]
  - [Dobrushin, Shlosman '87]
  - [Holley, Stroock '87, '89]
  - [Holley '91]
  - [Stroock, Zegarlinski '92a, '92b, '92c]
  - [Zegarlinski '90, '92]
  - [Lu, Yau '93]
  - [Martinelli, Olivieri '94a, '94b]
  - [Martinelli, Olivieri, Schonmann '94]
- $\Rightarrow$ Bounded log-Sobolev constant and $O(\log n)$ mixing.
- In two dimensions this is known for all $\beta < \beta_c$.

Mixing on the square lattice

- High temperature: gap^{-1} is uniformly bounded, $O(\log\ n)$ mixing for all $\beta < \beta_c = \frac{1}{2}\log(1+\sqrt{2})$.
  - Dynamics conjectured to exhibit *cutoff* [Peres'04].
  - Recently confirmed [L., Sly]: $t_{\text{mix}} = \frac{1+o(1)}{\lambda_\infty}\log n$
- Low temperature: for $\beta > \beta_c$ both gap^{-1} and the mixing time are $\exp[(c(\beta)+o(1))n]$.

  [Schonmann '87], [Chayes, Chayes, Schonmann'87], [Martinelli '94], [Cesi, Guadagni, Martinelli, Schonmann'96].
- Remains to verify power-law at critical $\beta = \beta_c$ …

?

Glauber dynamics at criticality

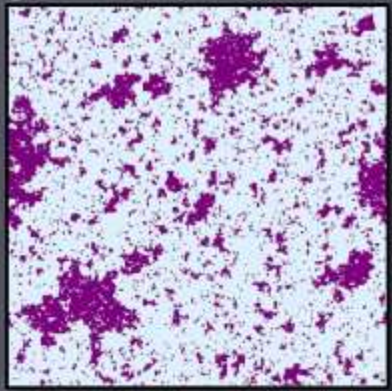
- Polynomial lower bound on gap^{-1} via the polynomial decay of spin-spin correlation whose asymptotics were established by [Onsager '44] ([cf. Holley '91]).
- Numerical experiments: $\exists$ universal exponent of ~ 2.17
  - [Ito '93], [Wang, Hatano, Suzuki '95], [Grassberger '95], [Nightingale, Blöte '96], [Wang, Hu '97],…
- Compared to conjectured power-law behavior of gap^{-1} :

? No known *sub-exponential* upper bounds …

- *Only geometries* with proved power-law for critical Ising:
  - Mean-field [Ding, L., Peres '09] (Curie-Weiss model)
  - Regular tree [Ding, L., Peres '10] (Bethe lattice).

Scaling limit of critical Ising

- Understanding of the limit developed emerged with the advent of SLE ([Schramm '00]), CLE and tools to study conformally invariant systems.
- Recent breakthrough results due to [Smirnov] describe full scaling limit of the Ising cluster interfaces as CLE with parameter $\kappa = 3$.



  - cf. [Werner '03], [Lawler-Werner '04], [Sheffield '09].
- Important role in the analysis of critical Ising: its counterpart Fortuin-Kasteleyn representation.

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Critical FK-Ising Model

- The FK-model is a measure over bond-percolation configurations also factoring in # of clusters.
- Scaling limits initially obtained for FK then converted to Ising.
  - E.g., full ensemble of FK cluster interfaces $\to \mathrm{CLE}_{16/3}$.
- Recent development via the above theory & tools:
  - Russo-Seymour-Welsh type estimates for FK-Ising with various BC due to [Duminil-Copin, Hongler, Nolin '09] [Camia, Newman '09], [Chelkak, Smirnov '09].

Main result: power-law at criticality

- THEOREM [L., Sly]: Critical slowdown verified in $\mathbb{Z}^2$:

Consider the critical Ising model on a finite box $\Lambda \subset \mathbb{Z}^2$ of side-length n, i.e. at inverse-temperature $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$. Let gap_Λ^τ denote the spectral-gap in the generator of the corresponding Glauber dynamics under an arbitrary fixed boundary condition τ. Then there exists an absolute $C > 0$ (independent of Λ, τ) such that $(\text{gap}_\Lambda^\tau)^{-1} \leq n^C$.

- COROLLARY:

Polynomial L^1 (total-variation) mixing time under any fixed boundary condition.

Further bounds on critical gap

- Analogous results for:
  - Free / periodic boundary conditions.
  - Critical anti-ferromagnetic Ising model.
- A new lower bound (previously known lower bound was nearly linear due to [Holley '91]).

THEOREM

Let $\mathrm{gap}_\Lambda^\tau$ denote the spectral-gap of the Glauber dynamics for critical Ising on a finite box $\Lambda \subset \mathbb{Z}^2$ of side-length n with an arbitrary boundary condition τ. Then $(\mathrm{gap}_\Lambda^\tau)^{-1} \geq cn^{7/4}$ for some absolute $c > 0$.

Ramifications for sampling

- First rigorous efficient algorithm for approximated sampling of critical 2D Ising (& its partition func.) for *arbitrary* (e.g. mixed) boundary conditions.
  - For the *free* boundary an efficient algorithm achieving this was given by [Jerrum Sinclair '93].
- Perfect simulation:
  - Enabled by the [Propp-Wilson '96] famous CFTP .
  - Applied to [JS '93] algorithm by [Randall Wilson '99] when boundary conditions are free/all-plus/all-minus.
  - New results allow *rigorous efficient perfect simulation* under any arbitrary boundary via CFTP for Glauber dynamics.

Main techniques

- Common approach for analyzing the dynamics:
  - Control rate of mixing via a spatial-mixing result for the influence of individual boundary-spins on distant sites.
  - Use decay of correlation with distance.
- At criticality (Onsager's work, also from "large" conformal loops with positive probability) there are long range correlations foiling this approach.
- Alternative approach:
  - Use conformal invariance to get a spatial-mixing result, combine it with classical ingredients from MC analysis.
  - Analyze effect of an *entire face* of the boundary on spins (just enough spatial mixing to push this program through…)

Generalized upper bound

- THEOREM

Consider critical Ising model on a box $\Lambda \subset \mathbb{Z}^2$ of dimensions $m \times n$. Let $\mathrm{gap}_\Lambda^\tau$ denote the spectral-gap in the generator of the corresponding Glauber dynamics under an arbitrary fixed boundary condition τ. There exists an absolute $C > 0$ (independent of Λ, τ) such that for any $m = m(n)$ we have

$$(\mathrm{gap}_\Lambda^\tau)^{-1} \le n^C.$$

- Only depends on the shorter side-length, e.g. on an extremely long rectangle of size $n \times \exp(\exp(n))$ we have the exact same n^C bound of given for the square.

Key spatial-mixing result

- THEOREM

Let $\Lambda = [\![1,r]\!] \times [\![1,r']\!]$ for some integers r, r' satisfying $r'/r \geq \alpha > 0$ with α fixed and let $\Lambda_{\mathrm{T}} = [\![1,r]\!] \times [\![\rho r, r']\!]$ for some ρ satisfying $\alpha \leq \rho < r'/r$. Let ξ, η be two BC's on Λ that differ only on the bottom boundary $[\![1,r]\!] \times \{0\}$. Then

$$\left\| \mu_{\Lambda}^{\xi}(\sigma(\Lambda_{\mathrm{T}}) \in \cdot) - \mu_{\Lambda}^{\eta}(\sigma(\Lambda_{\mathrm{T}}) \in \cdot) \right\|_{\mathrm{TV}} \leq \exp(-\delta\rho),$$

Where $\delta > 0$ is a constant that depends only on α.

- Proof uses the RSW-estimate for critical crossing probabilities in a wired FK-Ising rectangle.

Single site vs. Block dynamics

- Classical tool in the analysis of Glauber dynamics:
  - Cover the sites using blocks $\mathcal{B} = \{B_i\}$.
  - Each block updates via a rate-1 Poisson clock.
  - Updates are $\sim$ stationary given the rest of the system.

- <u>PROPOSITION</u> (see, e.g. [Martinelli '97]):

$$(\mathrm{gap}_\Lambda^\tau)^{-1} \leq \frac{\sum_\sigma \mu_\Lambda^\tau(\sigma) \sum_{x\in\Lambda} N_x c(x,\sigma)\left[f(\sigma^x) - f(\sigma)\right]^2}{\sum_\sigma \mu_\Lambda^\tau(\sigma) \sum_{x\in\Lambda} c(x,\sigma)\left[f(\sigma^x) - f(\sigma)\right]^2} (\mathrm{gap}_\mathcal{B}^\tau)^{-1} \max_{i,\varphi} (\mathrm{gap}_{B_i}^\varphi)^{-1}$$

where $(\mathrm{gap}_\mathcal{B}^\tau)^{-1}$ *is the gap of the block-dynamics and* $N_x = \#\{i : B_i \ni x\}$

Upper bound via spatial-mixing

- Consider the following choice of blocks:

$$\Lambda_1(\ell) = [\![1, r]\!] \times [\![\tfrac{1}{3} r' \tfrac{\ell-1}{3}\sqrt{rr'}, r']\!],$$
$$\Lambda_2(\ell) = [\![1, r]\!] \times [\![1, \tfrac{1}{3} r' + \tfrac{\ell}{3}\sqrt{rr'}]\!]$$

for some $\ell \in \{1,\dots,\lfloor \sqrt{r'/r} \rfloor\}$.

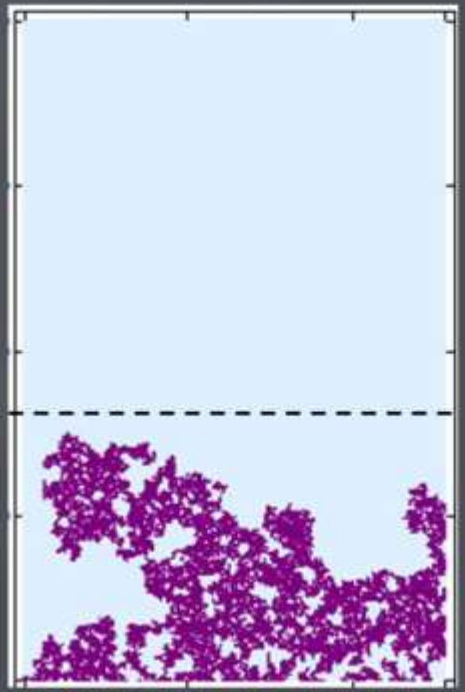
- The two blocks have a vertical overlap of height $\frac{1}{3}\sqrt{rr'}$.
- As a result of the spatial-mixing theorem:
For any boundary condition ξ on Λ we have $(\mathrm{gap}_{\mathcal{B}}^{\xi})^{-1} \geq 1 - \exp\left(-c\sqrt{r'/r}\right)$ for an absolute $c > 0$.

Upper bound via spatial-mixing (ctd.)

- Block dynamics reverts to a smaller block size at the cost of $1/[\,1-\exp(-c\sqrt{r'/r})\,]$.
- Average over the blocks to eliminate the contribution of N_x and replace it by $1+\frac{1}{\sqrt{r'/r}}$.
- Repeated applications yield $r' \le \frac{2}{3}r$ at the cost of an absolute constant.
- Iterating $\log_{3/2} n$ steps completes the proof. ■

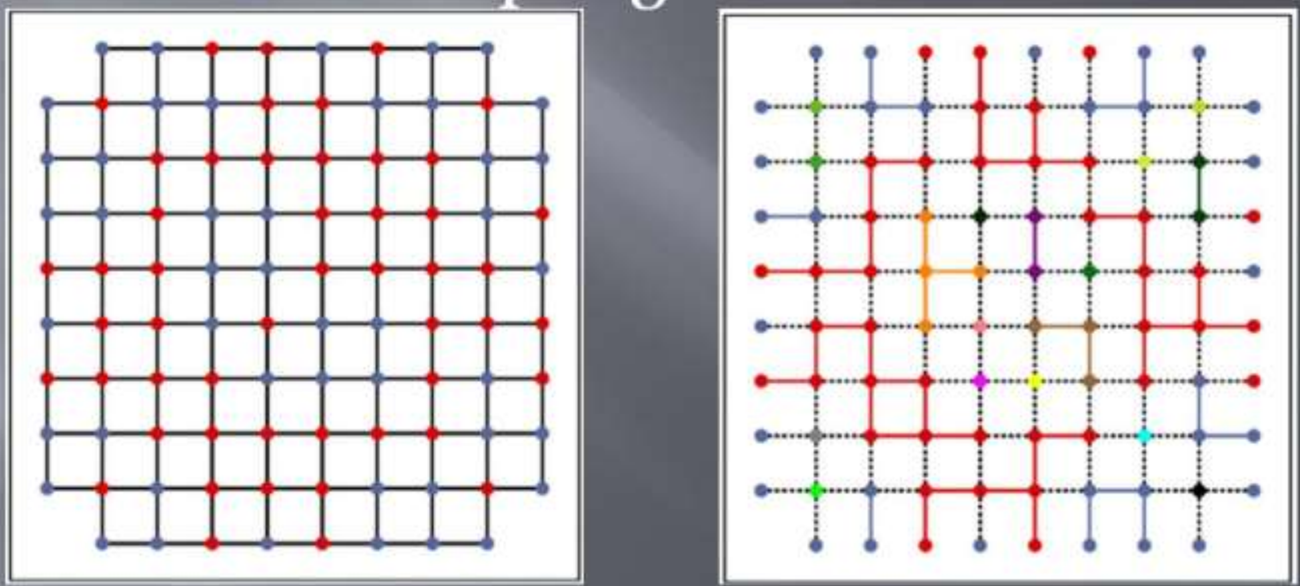
Intuition: spatial mixing proof

- Suppose that the three identical boundaries are all-minus, and the bottom boundary is all-plus in one measure and all-minus in the other.
  - Ising cluster adjacent to bottom in plus-measure converges to SLE_3, which does not climb past height $\rho\, r$ with positive probability.
  - In that case, measures can be coupled.
- Actual setting:
  - Arbitrary (mixed) boundary conditions break this argument down…



Solution: reduce to FK Ising

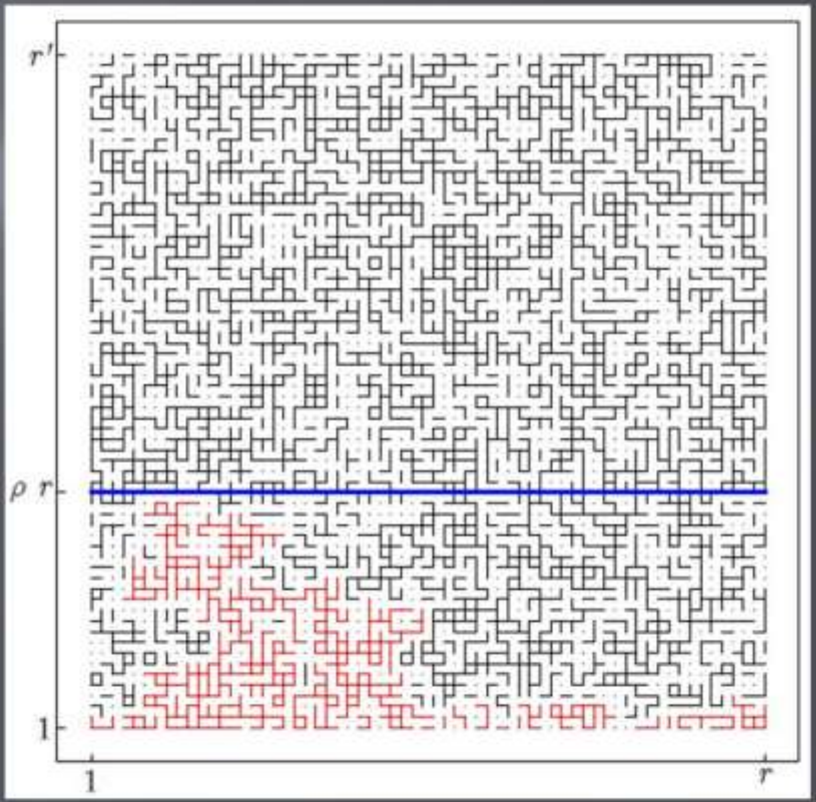
- Ising and its FK counterpart are coupled by the Edwards-Sokal coupling:



  - Under an arbitrary boundary condition ξ one can go from Ising $\rightsquigarrow$ FK $\rightsquigarrow$ Ising conditioned on some event A_ξ which may have exponentially small probability in FK…

Carrying the proof

- Control crossing probabilities in the FK-Ising model conditioned on the event A_ξ .
- Utilize the recent RSW-type estimates with the FKG for the FK-model to derive the required coupling.
- Return to Ising via the Edwards-Sokal method to complete the proof.



Open problems

- Calculate the precise (universal) critical dynamical exponent.
- Establish power-law behavior on the lattice in 3 dimensions.

THANK YOU.