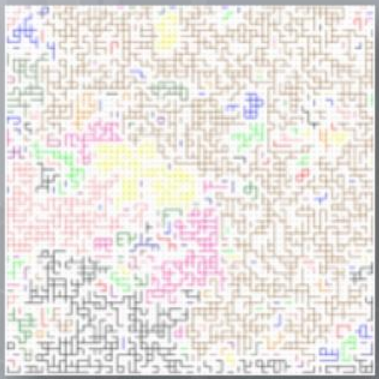
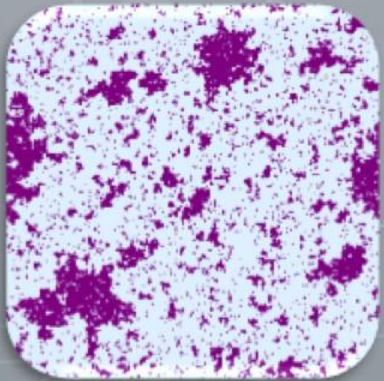
Harvard University

Applied Math Colloquium

Feb 2014

The Ising Model: Cutoff and Beyond





Eyal Lubetzky

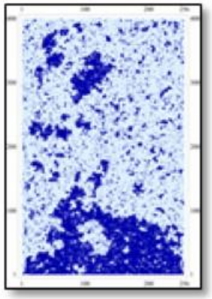
Microsoft Research

The Ising model

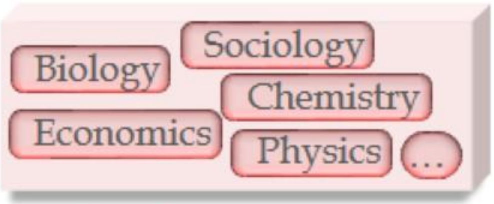

Wilhelm Lenz
1888–1957

- Introduced by Wilhelm Lenz in 1920 as a model of *ferromagnetism*:
  - Place iron in a magnetic field: increase field to maximum , then slowly reduce it to zero.
  - There is a critical temperature T_c (the Curie point) below which the iron retains residual magnetism.
- Magnetism caused by charged particles spinning or moving in orbit in alignment with each other.
- How do local interactions between nearby particles affect the global behavior at different temperatures?

The Ising model



- Gives random binary values (spins) to vertices accounting for nearest-neighbor interactions.
- Initially thought to be over-simplified to capture ferromagnetism, but turned out to have a crucial role in understanding phase transitions & critical phenomena.



- One of the most studied models in Math. Phys.: more than 10,000 papers over the last 25 years…

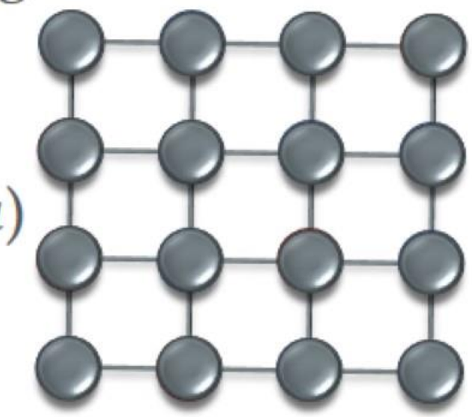
Ordered phase of short-range **Ising** spin-glasses
DS Fisher, DA Huse - Physical Review Letters, 1986 - APS
We propose a new picture of the **Ising**-spin-glass phase, based on an Ansatz for the scaling of low-lying large-scale-droplet excitations. We find behavior very different from the infinite-range model. The truncated spatial correlations decay as a power of distance, the ac nonlinear **...**
Cited by 375 - Related articles - All 3 versions

Definition: the classical Ising model

- Underlying geometry: Λ = finite 2D grid.
- Set of possible configurations:

$$\Omega = \{\pm 1\}^{\Lambda}$$

(each *site* receives a plus/minus *spin*)



- Probability of a configuration $\sigma \in \Omega$ given by the *Gibbs distribution*:

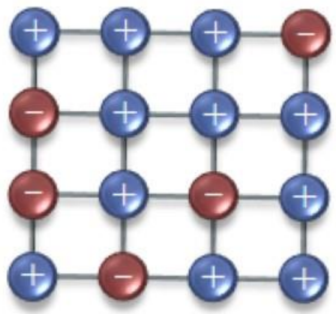
$$\mu(\sigma) = \frac{1}{Z(\beta)} \exp\left(\beta \sum_{x \sim y} \sigma(x)\sigma(y) \right)$$

Partition function

Inverse temperature $\beta \geq 0$

The classical Ising model

- $\mu(\sigma) \propto \exp\left(\beta \sum_{x\sim y} \sigma(x)\sigma(y)\right)$ for $\sigma \in \Omega = \{\pm 1\}^{\Lambda}$
  - Larger β favors configurations with aligned spins at neighboring sites.
  - Spin interactions: local, justified by rapid decay of magnetic force with distance.



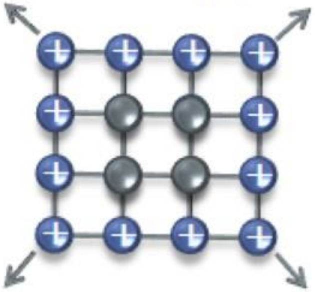
- The *magnetization* is the (normalized) sum of spins:

$$M(\sigma) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma(x)$$

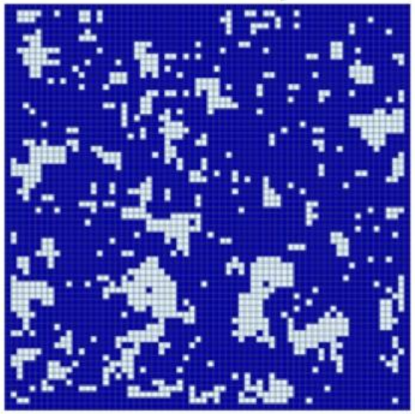
  - Distinguishes between disorder ($M \approx 0$) and order.
- Symmetry: $\mathbb{E}[M(\sigma)] = 0$. What if we *break the symmetry*?

The Ising phase–transition

- Ferromagnetism in this setting: [recall $M(\sigma) = \frac{1}{|\Lambda|}\sum \sigma(x)$]
  - Condition on the boundary sites all having *plus* spins.
  - Let the system size $|\Lambda|$ tend $\to \infty$ ($\approx$ a magnetic field with effect $\to 0$).

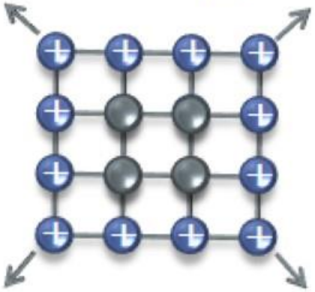


- What is the typical $M(\sigma)$ for large $|\Lambda|$? Does the effect of *plus* boundary vanish in the limit?



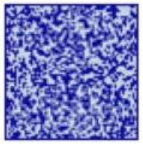
The Ising phase–transition (ctd.)

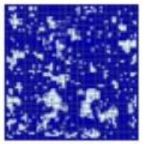
- Ferromagnetism in this setting: [recall $M(\sigma) = \frac{1}{|\Lambda|}\sum \sigma(x)$]
  - Condition on the boundary sites all having *plus* spins.
  - Let the system size $|\Lambda|$ tend $\to \infty$



- Expect: phase-transition at some critical β_c:

$$\lim_{|\Lambda|\to\infty} \mathbb{E}^{+}[M(\sigma)] = \begin{cases} 0 & \text{if } \beta < \beta_c \\ m_\beta > 0 & \text{if } \beta > \beta_c \end{cases}$$



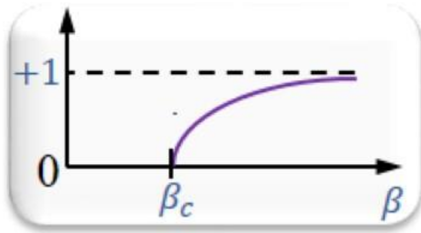


all-plus boundary

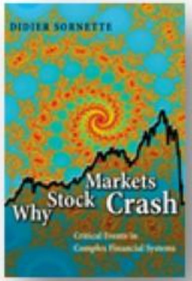
spontaneous magnetization

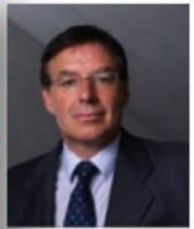
The Ising phase-transition (ctd.)

- The magnetization phase-transition at β_c :



- Replace *magnetization* ⇜⇝ *price* to find this diagram in "Why Stock Markets Crash" / D. Sornette (2001) [Chapter 5 "Modeling bubbles and crashes"]



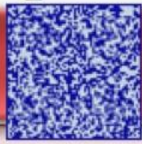


D. Sornette

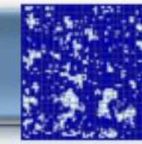
- Such applications of the Ising Model emphasize the dimension of ***time***:
  - How does the system evolve?
  - From a given starting state, how long does it take for certain configurations to appear?

Static vs. stochastic Ising

- Expected behavior for the Ising distribution:



$\beta < \beta_c$ | β_c | $\beta > \beta_c$



$\mathbb{E}^+[M(\sigma)] \xrightarrow[|\Lambda|\to\infty]{} 0$

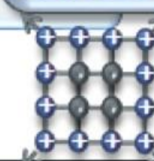
$\mathbb{E}^+[M(\sigma)] \xrightarrow[|\Lambda|\to\infty]{} c_\beta > 0$

- Expected behavior for the mixing time of dynamics:

$\beta < \beta_c$ | β_c | $\beta > \beta_c$

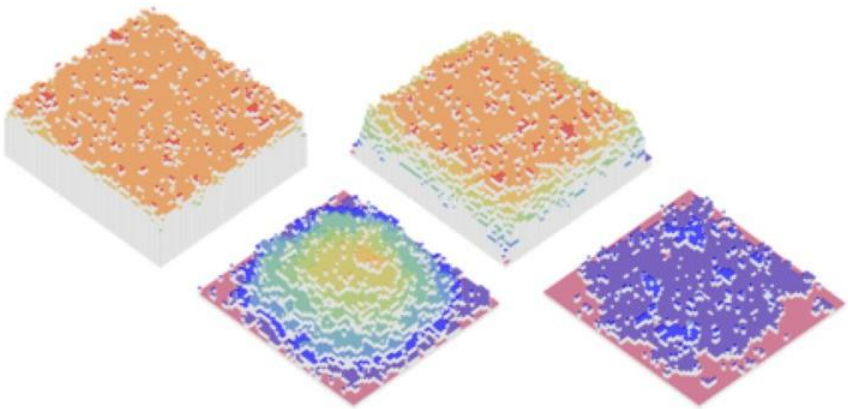
logarithmic | power law | exponential — *Free* b.c.

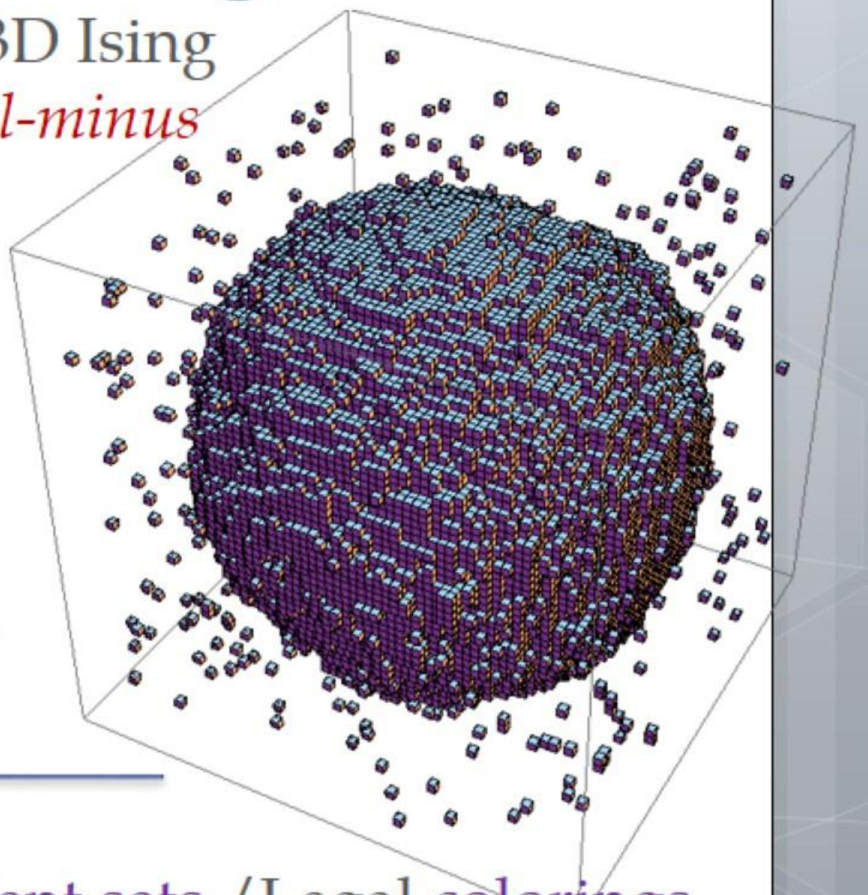
  - Symmetry breaking: power law — *Plus* b.c.



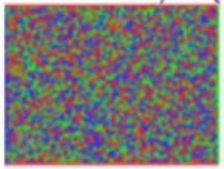
Example: stochastic Ising model

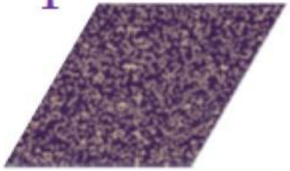
- Evolution of low-temperature 3D Ising with *all-plus* b.c. started from *all-minus*
- Related to models for crystals (*SOS, Discrete Gaussian*, etc.)

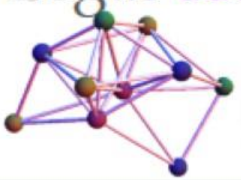




- One can ask also on:
Non-binary (Potts) / Independent sets / Legal colorings







More on this later…

The 1D Ising model

- Ph.D. in Physics in 1924 from U. Hamburg under the supervision of Lenz.
- Studied the 1D model of Lenz in his thesis:

Beitrag zur theorie des ferromagnetismus
E **Ising** - Zeitschrift für Physik A Hadrons and Nuclei, 1925 - Springer Cited by 2173

Ernst Ising
1900-1998

- Exact solution for the 1D model.
- Unfortunately: no phase-transition…
  - $\mathbb{E}^{+}[M(\sigma)] \xrightarrow[|\Lambda|\to\infty]{} 0$ for any $\beta \geq 0$.
  - Intuition: $(++++++++)$ similar to $(++++----)$
- Heuristic arguments why there would not be a phase-transition in higher dimensions either.

After solving the 1D model

- Ising [letter to S. Brush in 1967]:
  "...I discussed the result of my paper widely with Prof. Lenz and with Dr. Wolfgang Pauli, who at that time was teaching in Hamburg. There was some disappointment that the linear model did not show the expected ferromagnetic properties..."
- Left research after a few years at the German General Electric Co. and turned to teaching in public schools.
- Survived WW2 in Luxembourg isolated from scientific life. Came to the US in 1947 and only then "...did I learn that the idea had been expanded."

Meanwhile, on 2D Ising

- Heisenberg (1928) proposed his own theory of ferromagnetism, motivated by Ising's result.
- Followed by other models attempting to explain order/disorder in metallic alloys.
- In 1936 Rudolf Peierls published the paper

W. Heisenberg
1901-1976

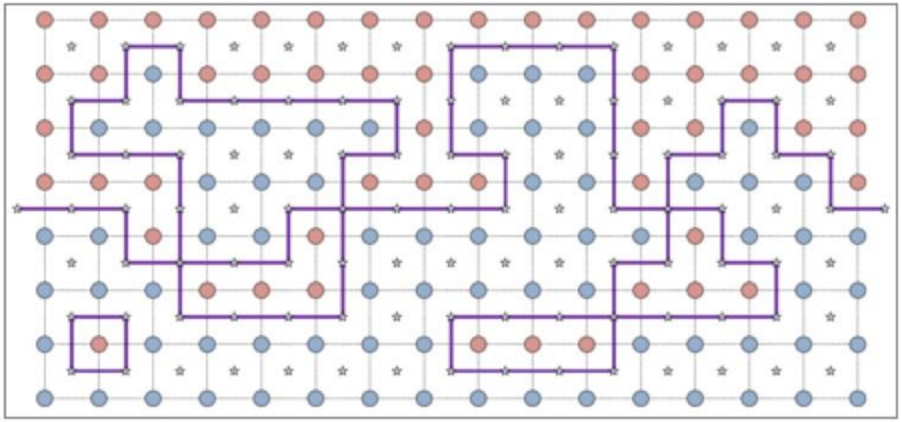
R. Peierls
1907-1995

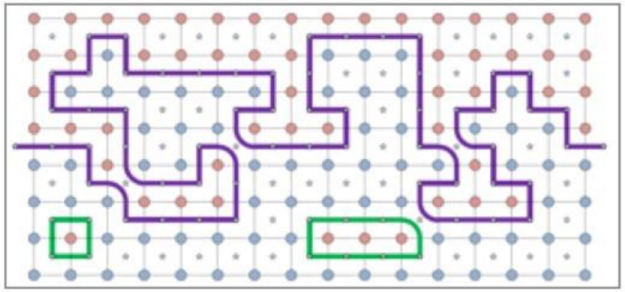
On **Ising's model** of **ferromagnetism**
R Peierls - Mathematical Proceedings of the Cambridge ..., 1936
Ising* discussed the following **model** of a **ferromagnetic** body: A
of moment yn to be arranged in a regular lattice; each of them is s
orientations, which we call positive and negative. Assume further t
Cited by 327 - Related articles - All 3 versions

arguing that the 2D and 3D Ising models *do* have spontaneous magnetization at *low enough temperature* (contrary to Ising's prediction).

Peierls' phase transition argument

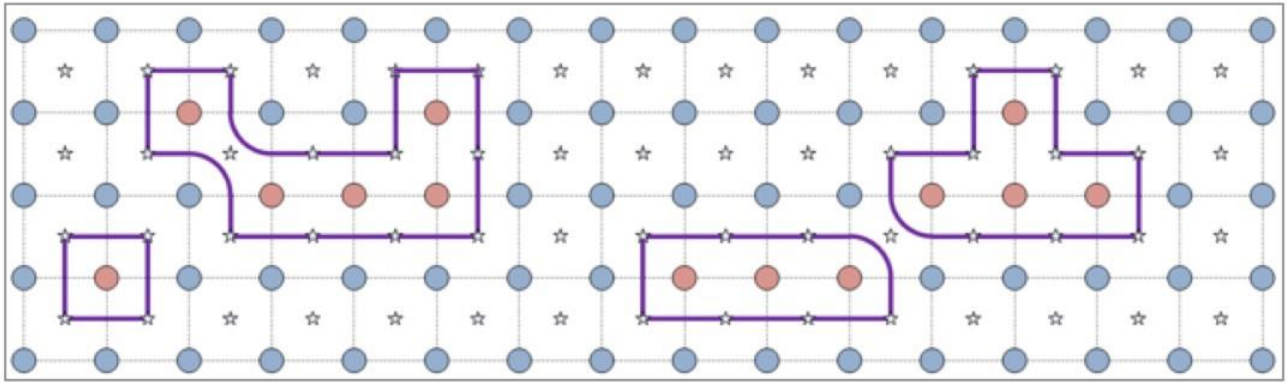
- Peierls' combinatorial argument is simple and robust.
- Key idea: represent Ising configurations as *contours* in the *dual graph*: the edges are dual to disagreeing edges.





Peierls' phase transition argument

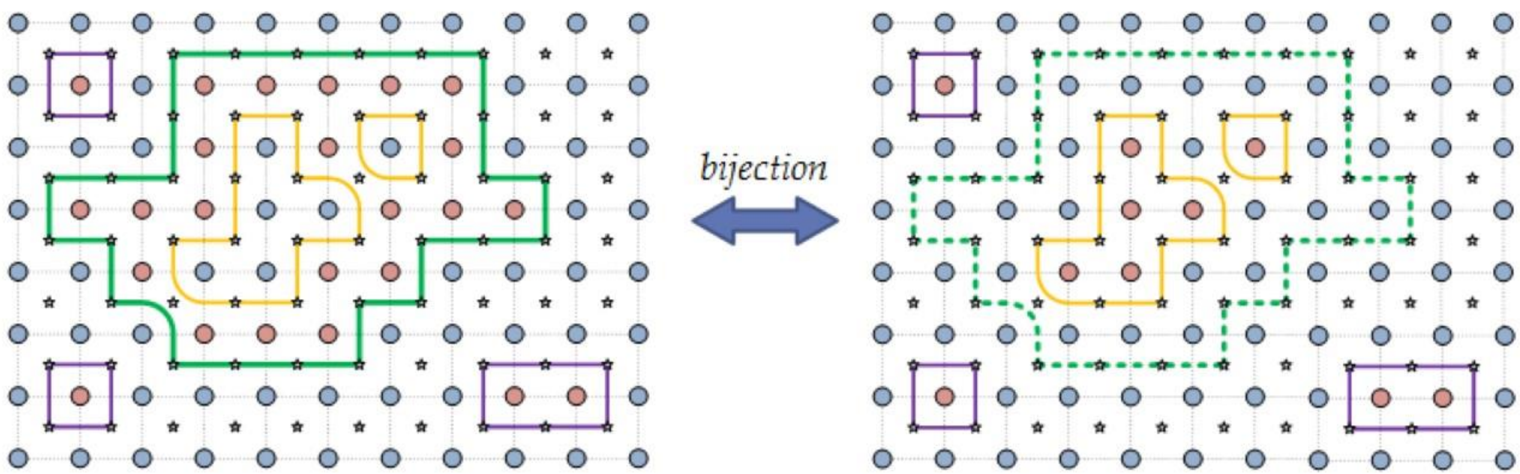
- When all boundary spins are + 's the Peierls contours are all closed [marking "islands" containing of – 's].



- The proof will follow a first moment argument on the number of sites inside such a – component.

Peierls' phase transition argument

- Setting: $\Lambda \subset \mathbb{Z}^2$ is an $n \times n$ box with all-plus boundary.
- Fix a contour C of length ℓ.
- For any σ containing C flip *all spins in the interior of* C :

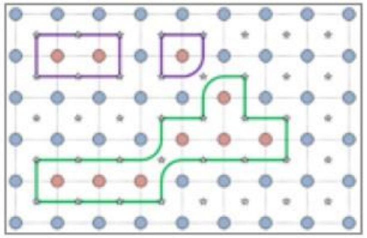


$$\mathbb{P}(\sigma) = \frac{1}{Z(\beta)} e^{B-\beta\ell} \qquad \mathbb{P}(\sigma') = \frac{1}{Z(\beta)} e^{B+\beta\ell}$$

- $\Rightarrow \mathbb{P}(C \text{ belongs to contours}) \leq e^{-2\beta\ell}$.

Peierls' phase transition argument

- For a fixed contour C of length ℓ :
  - $\Rightarrow \mathbb{P}(C \text{ belongs to contours}) \le e^{-2\beta\ell}$.
  - C can contain $\le \ell^2$ sites (isoperimetric).



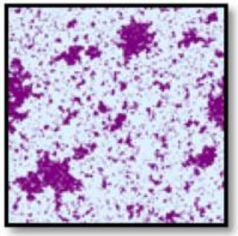
- $O(N\,3^\ell)$ possible such contours, where $N = |\Lambda| = n^2$.

- Summing we get:

$$\mathbb{E}[\#\{x : \sigma(x) = -1\}] \lesssim N \sum_\ell \ell^2 \left(3e^{-2\beta}\right)^\ell < \varepsilon N$$

where $\varepsilon < 1/2$ for a suitably large β. ∎

Landmarks for 2D Ising



- Critical point candidate $\beta_c = \frac{1}{2}\log(1+\sqrt{2}) \approx 0.44$ found by Kramers and Wannier in 1941 via duality.
- 2D Ising model *exactly solved* in 1944 in seminal work of Lars Onsager (Nobel in Chemistry 1968).
  - Proof analyzed the $2^n \times 2^n$ transfer matrix using the theory of Lie algebras.

L. Onsager
1903-1976

- Understanding of critical geometry boosted by advent of SLE [Schramm '00] and breakthrough results of Smirnov.

O. Schramm
1961-2008

S. Smirnov

- In parallel: extensive study of the dynamical model...

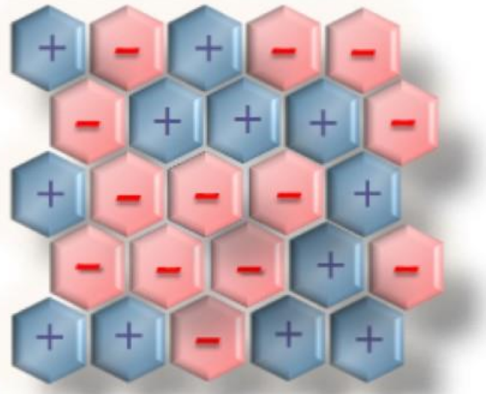
Glauber dynamics for Ising

(*a.k.a.* the *Stochastic Ising model*)

R.J. Glauber

- MCMC sampler introduced in 1963 by Roy Glauber (Nobel in Physics 2005).

Time-dependent statistics of the Ising model
RJ Glauber – Journal of mathematical physics, 1963 Cited by 2749



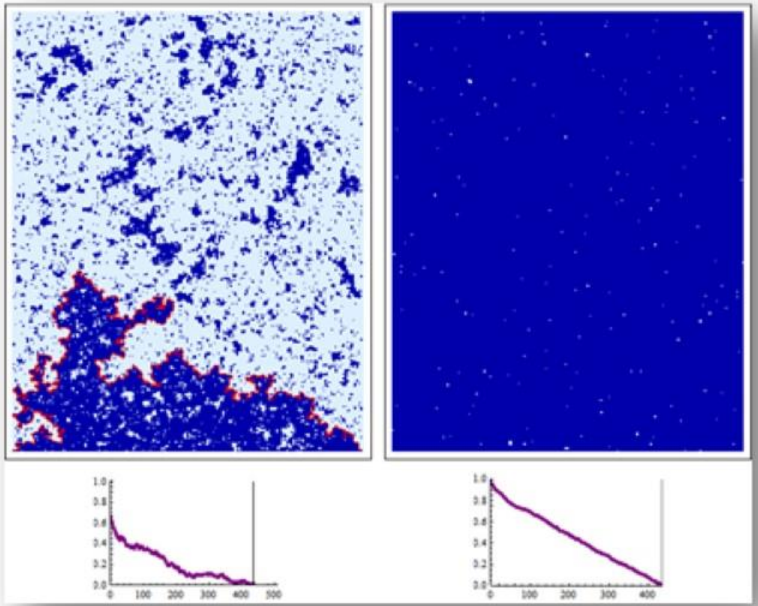
- One of the most commonly used samplers for the Ising distribution μ:
  - Update sites via IID Poisson(1) clocks
  - Each update replaces a spin at $x \in V$ by a new spin $\sim \mu$ given spins at $V \setminus \{x\}$.

  (*heat-bath* version; famous other flavor: *Metropolis*)

- How long does it take it to converge to μ ?

$\frac{1}{Z(\beta)} e^{B-2\beta}$ $\frac{1}{Z(\beta)} e^{B+2\beta}$

Example: Glauber dynamics for the Ising model on a square lattice



- 256×320 square lattice
- Frame every 2^{23} steps (~100 updates per site).

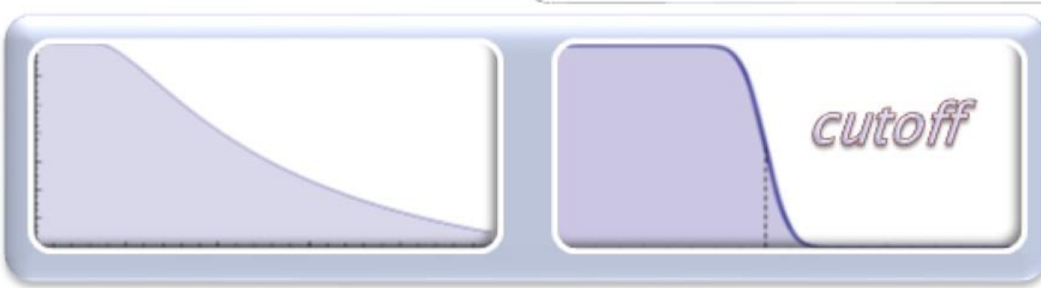
Measuring convergence to equilibrium

- Mixing time : (according to a given metric).
  Standard choice: L^1 (total-variation) mixing time to within distance ε is defined as

$$\boxed{t_{\mathrm{mix}}(\varepsilon) = \inf\left\{t \ : \max_{x_0} \|p^t(x_0,\cdot) - \mu\|_{\mathrm{tv}} \le \varepsilon\right\}}$$

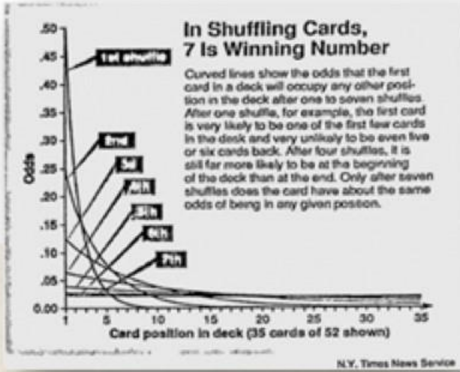
(where $\|\mu - \nu\|_{\mathrm{tv}} = \sup_{A\subset\Omega}[\mu(A) - \nu(A)]$)

- Dependence on ε : (*the cutoff phenomenon*)
  We say there is ***cutoff*** $\Leftrightarrow$ $\boxed{t_{\mathrm{mix}}(\varepsilon) \sim t_{\mathrm{mix}}(\varepsilon') \quad \forall \text{ fixed } \varepsilon, \varepsilon'}$



'7 shuffles suffice'

The New York Times January 9, 1990



It takes just seven ordinary, imperfect shuffles to mix a deck of cards thoroughly, researchers have found. Fewer are not enough and more do not significantly improve the mixing.

… Dr. Persi Diaconis, a mathematician and statistician at Harvard University who is the other author of the discovery, said the methods used are already helping mathematicians analyze problems in abstract mathematics that have nothing to do with shuffling…

… By saying that the deck is completely mixed after seven shuffles, Dr. Diaconis and Dr. Bayer mean that every arrangement of the 52 cards is equally likely or that any card is as likely to be in one place as in another.

The cards do get more and more randomly mixed if a person keeps on shuffling more than seven times, but seven shuffles is a transition point, the first time that randomness is close. Additional shuffles do not appreciably alter things…

The Cutoff Phenomenon


D. Aldous


P. Diaconis

- Discovered in [DS'81], [A'83], [AD'86].
- Nearly 3 decades after its discovery:
  no example of cutoff for RW on ***bounded-degree graph***

- <u>CONJECTURE</u> (Durrett '07):
  cutoff for RW on almost every 3-regular graph.
- Confirmed in [L., Sly '10]:

SRW on almost every n-vertex d-regular graph ($d \geq 3$ fixed) has cutoff at $\frac{d}{d-2} \log_{d-1} n$ with window $O\left(\sqrt{\log n}\right)$.

- Mysterious even for uniform stationary measure:

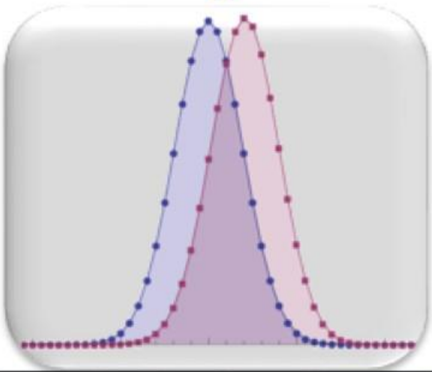
  (conj. of Peres)

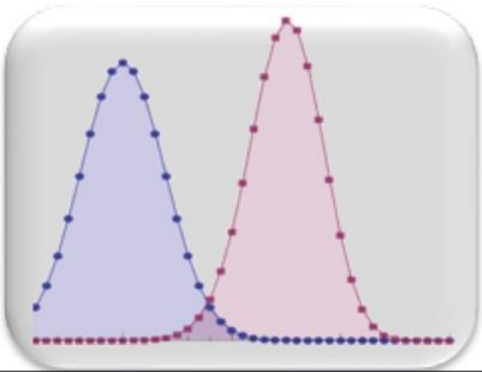
  - ? cutoff for RW on ***every*** transitive 3-regular **expander**?
  - ? cutoff for RW on ***any*** transitive 3-regular **expander**?

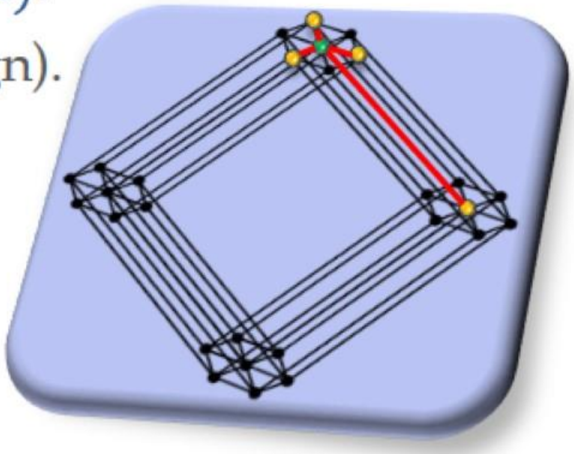
RW on the hypercube

Discrete analog: uniformly choose a site $x \in \{1, \dots, n\}$ and a new $\{0,1\}$ value for it

- Equivalent to the Glauber dynamics at $\beta = 0$.
- Straightforward bounds: $\frac{1}{2}\log n \le t_{\mathrm{mix}} \le \log n$.
- [Aldous '83]: cutoff: $t_{\mathrm{mix}}(\varepsilon) = \frac{1}{2}\log n + O(1)$
  - Symmetry: start at the all-1 state.
  - # of 1's at time t is $\sim \mathrm{Bin}(n, (1+e^{-t})/2)$.
  - # of 1's under stationary measure $\sim$ $\mathrm{Bin}(n, ½)$, which has Gaussian fluctuations of $O(\sqrt{n})$.
  - Mixing when $e^{-t} \asymp \sqrt{n}$ (fluctuations align).







Believed picture for Ising on $\mathbb{Z}_n^d$

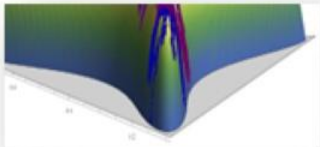
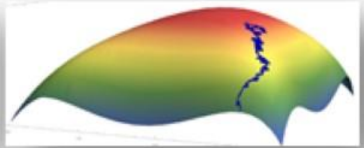
- For some critical inverse-temperature β_c:

$\beta < \beta_c$ | β_c | $\beta > \beta_c$ | free b.c.

t_{mix}: $\sim c_\beta \log n$ | $\asymp n^z$ | $\asymp e^{\tau_\beta\, n^{d-1}}$

Cutoff ! | universal

- Analogous picture verified for:
  - Complete graph [Ding, L., Peres '09a, '09b], [Levin, Luczak, Peres '10] :
    $\frac{1}{2(1-\beta)}\log n + O(1)$ | $\asymp \sqrt{n}$ | $\asymp \frac{1}{\beta-1}\exp\left[\frac{3}{4}(\beta-1)^2 n\right]$
  - Regular tree [Berger, Kenyon, Mossel, Peres '05] (high T/low T)
    [Ding, L., Peres '10] (critical T)
  - Potts model on complete graph
    [Cuff, Ding, L., Louidor, Peres, Sly '12]

Glauber dynamics for 2D Ising

- Fast mixing at **high** temperatures:
  - [Aizenman, Holley '84]
  - [Dobrushin, Shlosman '87]
  - [Holley, Stroock '87, '89]
  - [Holley '91]
  - [Stroock, Zegarlinski '92a, '92b, '92c]
  - [Lu, Yau '93]
  - [Martinelli, Olivieri '94a, '94b]
  - [Martinelli, Olivieri, Schonmann '94]

$\beta < \beta_c$

$t_{\text{mix}} \asymp \log n$

- Slow mixing at **low** temperatures:
  - [Schonmann '87]
  - [Chayes, Chayes, Schonmann '87]
  - [Martinelli '94]
  - [Cesi, Guadagni, Martinelli, Schonmann '96]

$\beta > \beta_c$ *free* b.c.

$t_{\text{mix}} = e^{(\tau_\beta + o(1))n^{d-1}}$

- **Critical** power-law?
  - *lower bound*: [Aizenman, Holley '84], [Holley '91]
  - *simulations*: [Ito '93], [Wang, Hatano, Suzuki '95], [Grassberger '95], [Nightingale, Blöte '96], [Wang, Hu '97],…

β_c

no sub-exponential upper bounds

$t_{\text{mix}} \geq n^c$
sim: $n^{2.17\ldots}$

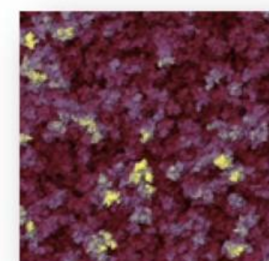
Glauber dynamics for 2D Ising

	$\beta < \beta_c$	β_c	$\beta > \beta_c$ (free b.c.)
t_{mix}:	$O(\log n)$	$\geq n^c$ sim: $n^{2.17\ldots}$	$e^{(\tau_\beta + o(1))n^{d-1}}$

- Recent progress:
  - [L., Sly '13]: ***cutoff*** for any $\beta < \beta_c$:

    $$t_{\text{mix}}(\varepsilon) = \tfrac{1}{2}\lambda_\infty^{-1} \log n + O(\log\log n)$$

    - analysis via log-Sobolev inequalities & new method to break long-range dependencies.



  - [L., Sly '12]: ***power-law*** at β_c:

    $$t_{\text{mix}} = O(n^C)$$

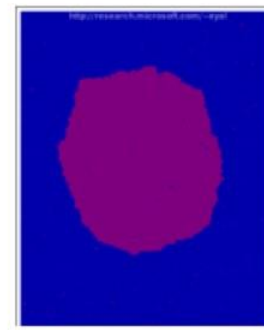
    - analysis used SLE behavior of critical interfaces.

Glauber dynamics for 2D Ising

$\beta > \beta_c$ *Plus b.c.*

- Mixing at *low temperatures* under *all-plus* boundary:
  - <u>CONJECTURE</u>: (Fisher, Huse '87) $t_{\text{mix}} \asymp n^2$.
  - [Martinelli '94]: $t_{\text{mix}} \le \exp\left(n^{1/2+o(1)}\right)$ $\beta \gg \beta_c$
  - [Martinelli, Toninelli '11]: $t_{\text{mix}} \le \exp\left(n^{o(1)}\right)$ $\beta \gg \beta_c$

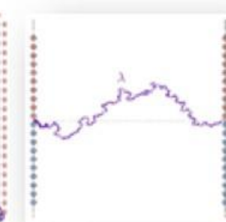
power law?



- Recent progress: *quasi-polynomial* mixing
  - [L., Martinelli, Toninelli, Sly '13]:

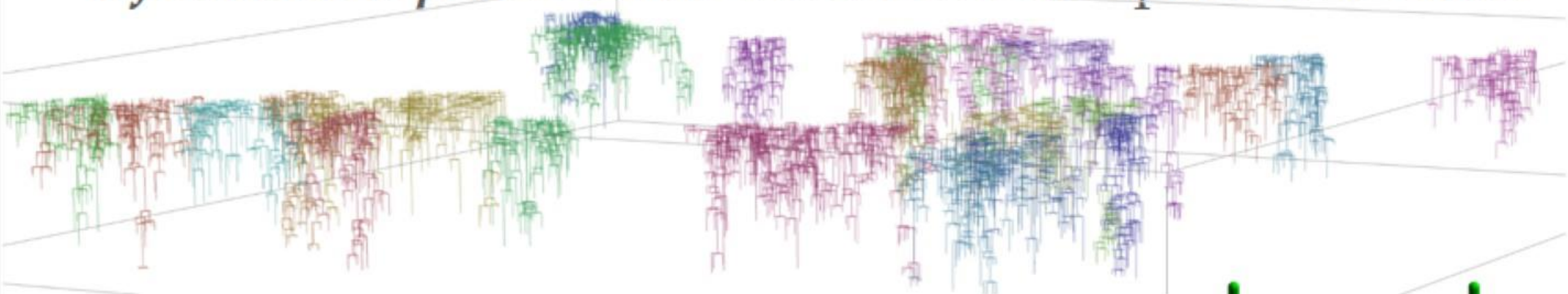
    $\boxed{t_{\text{mix}} = n^{O(\log n)}}$ for any $\beta > \beta_c$.

    - uses interface convergence to Brownian bridges

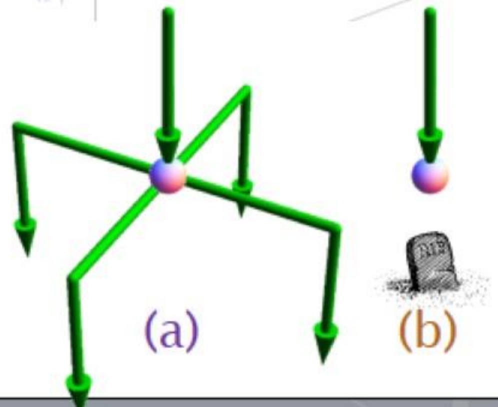


Progress at high temperature

- Traditional approach to sharp mixing results
  1. Establish spatial properties of static Ising measure
  2. Use to drive a multi-scale analysis of dynamics.
- New approach: study these *simultaneously* examining ***information percolation*** clusters in the space-time slab:

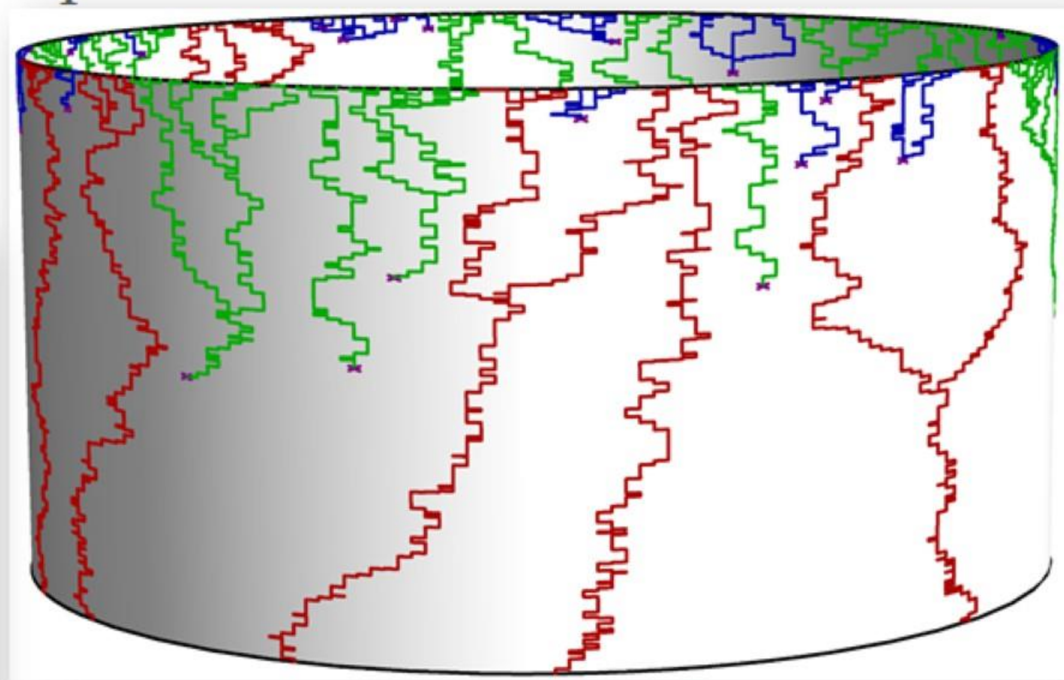


  - track update lineage back in time.
  - update either (a) branches out, or (b) terminates ("*oblivious*")

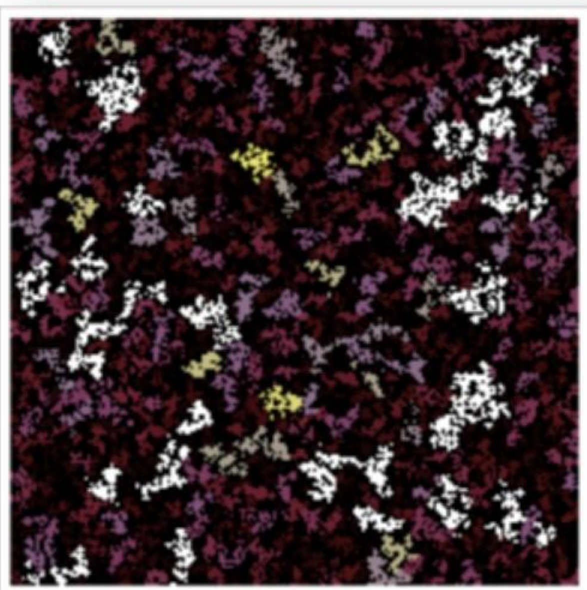


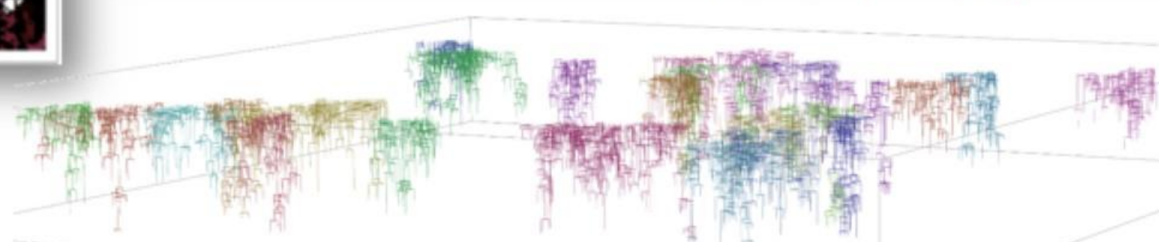
The framework

- R/G/B information percolation clusters:



the 3 cluster classes (R/G/B) in $\mathbb{Z}_{256}$

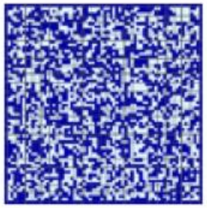


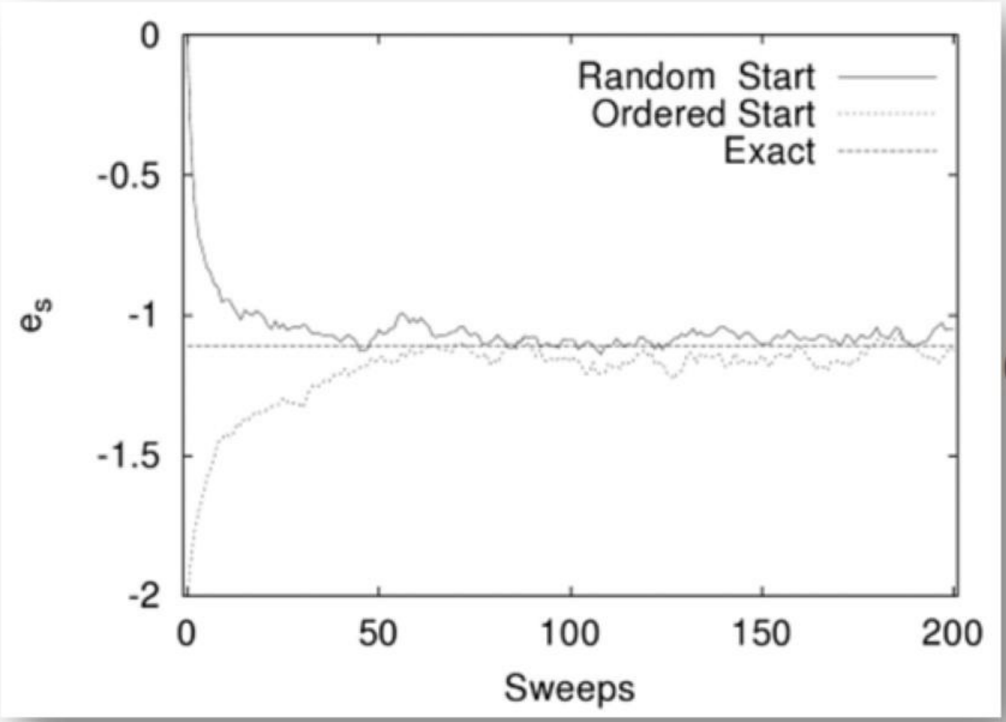


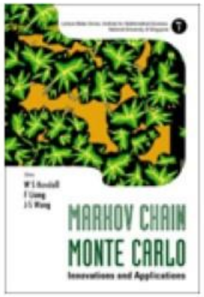
$\mathbb{Z}_{200}^2$ cluster
(top/side view)

High temperature paradigms?

- Frequent element in MCMC literature:
  random (disordered) vs. cold (ordered) starting states





(Chapter 1 in 

)

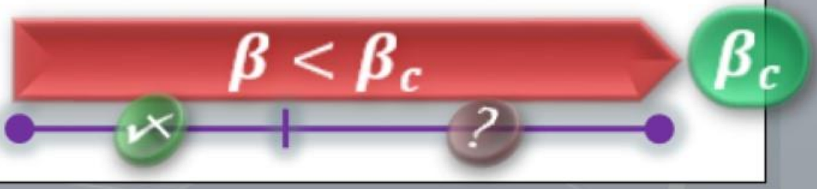
High temperature paradigms?

I. *Warm start (random, disordered) vs. cold start (ordered):*
Does this accelerate mixing? if so, by how much?

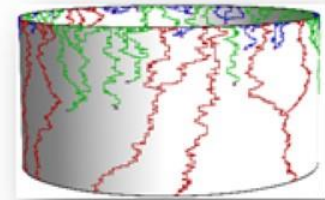
- *previously*: no sharp rigorous bounds for mixing on non-worst-case starting states.

II. *High temperature vs. Infinite temperature:*
Qualitatively, $\beta < \beta_c$ believed to behave $\approx$ as $\beta = 0$.

- $\beta = 0$: cutoff ([Aldous '83]): $t_{\text{mix}}(\varepsilon) = \frac{1}{2}\log n + O(1)$.
- $\Rightarrow$ expect cutoff for all $\beta < \beta_c$ (conjectured by [Peres '04]) and furthermore, with an $O(1)$-window.
- *previously*: full range $\beta < \beta_c$ covered just for dim $d \leq 2$, and only with a $O(\log\log n)$-window.

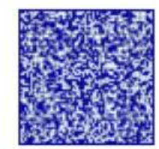


New results: initial states



- Example: the 1D Ising model ($\mathbb{Z}_n$):
  - All-plus starting state is worst (up to an additive $O(1)$) [but twice faster than standard monotone coupling bound].

  - Uniform starting distribution is asymptotically twice faster than the worst-case all-plus.



  - Almost all deterministic initial states are asymptotically as bad as the worst-case all-plus.
- <u>THEOREM:</u> ([L.-Sly '14+])

Fix $\beta > 0$ and $0 < \varepsilon < 1$; set $t_{\mathrm{m}} = \frac{1}{2(1-\tanh(2\beta))} \log n$.

1. (*Annealed*) $t_{\mathrm{mix}}^{(\mathrm{U})}(\varepsilon) \sim \frac{1}{2} t_{\mathrm{m}}$
2. (*Quenched*) $t_{\mathrm{mix}}^{(x_0)}(\varepsilon) \sim t_{\mathrm{mix}}^{(+)}(\varepsilon) \sim t_{\mathrm{m}}$ for almost $\forall x_0$

New results: high–temp ≈ ∞–temp

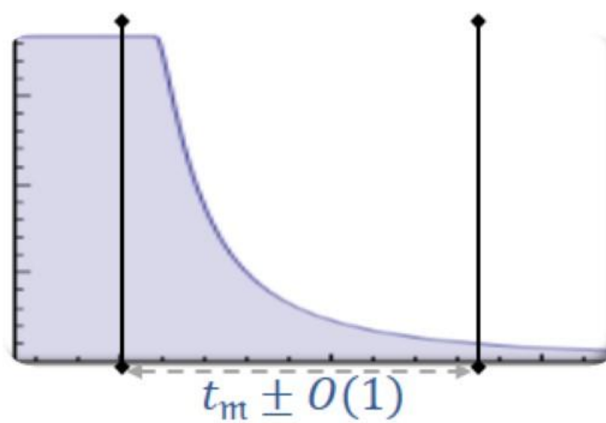
- Confirm Peres's conj. on $\mathbb{Z}_n^d$ for any d, with $O(1)$-window.
- <u>THEOREM:</u> ([L.-Sly '14+])

> $\forall d \geq 1$ and $\beta < \beta_c$ there is cutoff with an $O(1)$-window at
> $$t_{\mathrm{m}} = \inf\left\{t : \mathbb{E}^+ M(\sigma_t) \leq \sqrt{n^d}\right\}$$

cutoff window: $O(\log(1/\varepsilon))$

[recall $M(\sigma) = \frac{1}{|\Lambda|}\sum \sigma(x)$]

- Examples:
  - $d = 1$: $t_{\mathrm{m}} = \frac{1}{2(1-\tanh(2\beta))}\log n$.
  - $\beta = 0$: $t_{\mathrm{m}} = \frac{1}{2}\log n$ (matching [Aldous '83])



New results: universality of cutoff

- **Paradigm:** cutoff characterizes high temperature *independently of the geometry.*
  - ($\exists$ cutoff for Ising on an expander ?)

$\beta < \beta_c$ → Cutoff

- **Universality of cutoff**: analogous results for *any locally finite geometry* at high enough temperature!
- THEOREM: ([L.-Sly '14+]

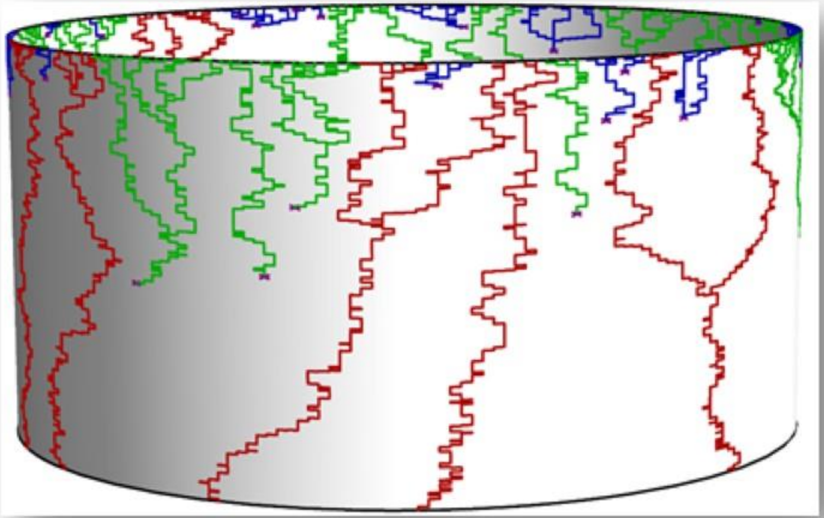
$\exists\, \kappa > 0$ so that, if G is any n-vertex graph with degrees $\leq d$ and $\beta < \kappa/d$, then $\exists$ cutoff with an $O(1)$-window at

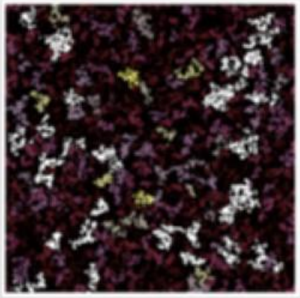
$$t_{\mathrm{m}} = \inf\left\{t : \textstyle\sum_x \mathbb{E}^+\left[M\big(\sigma_t(x)\big)^2\right] \leq 1\right\}.$$

- Similarly: $t_{\mathrm{mix}}^{(\mathrm{U})} \sim \left(\frac{1}{2} + \delta_\beta\right) t_{\mathrm{m}}$ yet $t_{\mathrm{mix}}^{(x_0)} \geq (1 - \delta_\beta) t_m$ a.e. x_0.

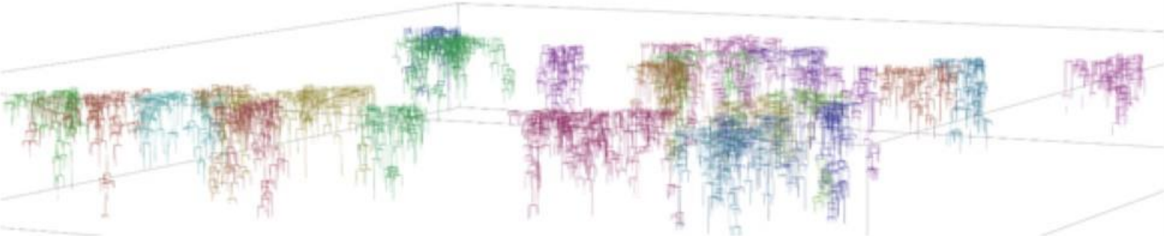
The framework (revisited)

- R/G/B information percolation clusters:
  - In 1D: $\theta = \mathbb{P}(\text{oblivious update}) = 1 - \tanh 2\beta$
  - Update history: continuous-time RW killed at rate θ.
  - $\mathbb{P}(\text{surviving to time } t_{\mathrm{m}})$ is $\asymp 1/\sqrt{n}$.



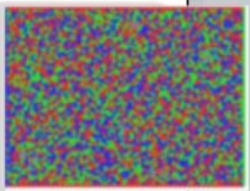


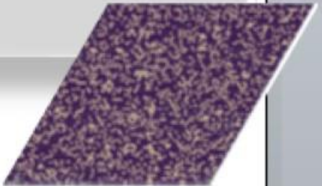
the 3 cluster classes (R/G/B) in $\mathbb{Z}_{256}$



General spin–systems

- [L.-Sly '14+]: cutoff for general spin-systems on $\mathbb{Z}_n^d$, e.g.:
  - Proper coloring with $q \geq 4d(d+1)$ colors
  - Potts model: $q \geq 2$ colors and $1 \leq \lambda < \left(1+\frac{q}{2d}\right)^{-4d}$
  - Independent sets, Anti-ferromagnetic Potts,…

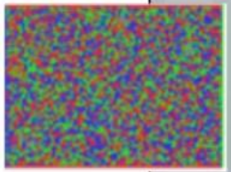




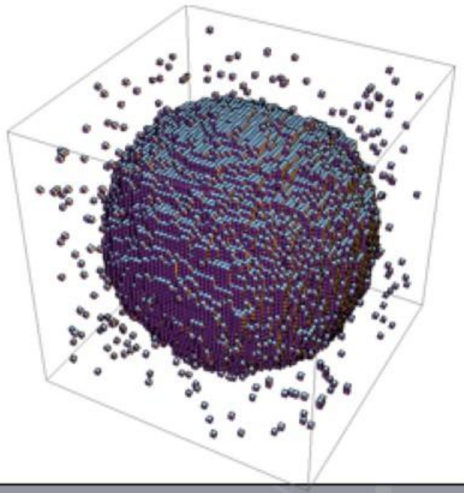
  Caveats:
  - needs *sub-exponential growth* & *very high temperature*;
  - non-explicit cutoff location, suboptimal cutoff window.
- Recent cutoff results for spin systems in 1D:
  - [Lacoin '14+]: Exclusion process
  - [Ganguly, L., Martinelli '14+]: East process

Open problems

- High temperature regime for spin-systems on lattices (*e.g.*, Potts / Independent sets /Legal colorings):
  - asymptotic mixing in full high temperature regime
  - cutoff for colorings on a transitive **expander**
  - asymptotic mixing from uniform starting state (*e.g.*, compare ordered/disordered start in Potts)



- 3D Ising:
  - power-law behavior at criticality
  - sub-exponential upper bound at low temperatures under all-plus b.c.



Thank you