DYNAMICS OF $2+1$ DIMENSIONAL SOS SURFACES ABOVE A WALL: SLOW MIXING INDUCED BY ENTROPIC REPULSION

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Abstract. We study the Glauber dynamics for the (2+1)D Solid-On-Solid model above a hard wall and below a far away ceiling, on an $L \times L$ box of $\mathbb{Z}^2$ with zero boundary conditions, at large inverse-temperature β. It was shown by Bricmont, El-Mellouki and Fröhlich [12] that the floor constraint induces an entropic repulsion effect which lifts the surface to an average height $H \asymp (1/\beta)\log L$. As an essential step in understanding the effect of entropic repulsion on the Glauber dynamics we determine the equilibrium height H to within an additive constant: $H = (1/4\beta)\log L + O(1)$. We then show that starting from zero initial conditions the surface rises to its final height H through a sequence of metastable transitions between consecutive levels. The time for a transition from height $h = aH$, $a \in (0,1)$, to height $h+1$ is roughly $\exp(cL^a)$ for some constant $c > 0$. In particular, the mixing time of the dynamics is exponentially large in L, i.e. $T_{\textsc{mix}} \geqslant e^{cL}$. We also provide the matching upper bound $T_{\textsc{mix}} \leqslant e^{c'L}$, requiring a challenging analysis of the statistics of height contours at low temperature and new coupling ideas and techniques. Finally, to emphasize the role of entropic repulsion we show that without a floor constraint at height zero the mixing time is no longer exponentially large in L.

1. Introduction

The $(d+1)$-dimensional *Solid-On-Solid* model is a crystal surface model whose definition goes back to Temperley [51] in 1952 (also known as the Onsager-Temperley sheet). Its configuration space on a finite box $\Lambda \subset \mathbb{Z}^d$ with a floor (wall) at 0, a ceiling at some n^+ and zero boundary conditions is the set Ω_{Λ,n^+} of all height functions η on $\mathbb{Z}^d$ such that $\Lambda \ni x \mapsto \eta_x \in \{0, 1, \ldots, n^+\}$ whereas $\eta_x = 0$ for all $x \notin \Lambda$. The probability of $\eta \in \Omega_{\Lambda,n^+}$ is given by the Gibbs distribution

$$\pi_\Lambda(\eta) = \frac{1}{Z_\Lambda} \exp\Bigg(-\beta \sum_{x \sim y} |\eta_x - \eta_y| \Bigg), \tag{1.1}$$

where $\beta > 0$ is the inverse-temperature, $x \sim y$ denotes a nearest-neighbor bond in the lattice $\mathbb{Z}^d$ and the normalizing constant Z_Λ is the partition-function.

Numerous works have studied the rich random surface phenomena, e.g. roughening, localization/delocalization, layering and wetting to name but a few, exhibited by the SOS model and some of its many variants. These include the *discrete Gaussian* (replacing $|\eta_x - \eta_y|$ by $|\eta_x - \eta_y|^2$ for the integer analogue of the Gaussian free field), *restricted* SOS (nearest neighbor gradients restricted to $\{0, \pm 1\}$), *body centered* SOS [6], etc. (for more on these flavors see e.g. [1,4,10]).

Of special importance is SOS with $d = 2$, the only dimension featuring a *roughening transition*. Consider the SOS model without constraining walls (the height function η takes values in $\mathbb{Z}$). For $d = 1$, it is well known ([25,51,52]) that the SOS surface is *rough* (delocalized) for any $\beta > 0$, i.e., the expected height at the origin (in absolute value) diverges in the thermodynamic limit $|\Lambda| \to \infty$. However, for $d \geqslant 3$ a Peierls argument shows that the surface is *rigid* (localized) for any $\beta > 0$ (see [13]), i.e., $|\eta_0|$ is uniformly bounded in expectation. This is also the case for $d = 2$

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and large enough β ([11, 32]). That the surface is rough for $d = 2$ at high temperatures was established in seminal works of Fröhlich and Spencer [29–31]. Numerical estimates for the critical inverse-temperature β_{R} where the roughening transition takes place suggest that $\beta_{\mathrm{R}} \approx 0.806$.

One of the main motivations for studying an SOS surface constrained between two walls, both its statics and its dynamics, stems from its correspondence with the Ising model in the phase coexistence region. For concreteness, take a box of side-length L in $\mathbb{Z}^3$ with minus boundary conditions on the bottom face and plus elsewhere. One can view the $(2+1)$D SOS surface taking values in $\{0,\dots,L\}$ as the interface of the minus component incident to the bottom face, in which case the Hamiltonian in (1.1) agrees with that of Ising up to bubbles in the bulk. At low enough temperatures bubbles and interface overhangs are microscopic, thus SOS should give a qualitatively correct approximation of Ising (see [3, 25, 48]). Indeed, in line with the (2+1)D SOS picture, it is known [5] that the 3D Ising model undergoes a roughening transition at some $\beta_{\mathrm{R}}^{\mathrm{IS}}$ satisfying $\beta_c(3) \leqslant \beta_{\mathrm{R}}^{\mathrm{IS}} \leqslant \beta_c(2)$ (where $\beta_c(d)$ is the critical point for Ising on $\mathbb{Z}^d$), yet there is still no rigorous proof that $\beta_{\mathrm{R}}^{\mathrm{IS}} > \beta_c(3)$ (see [1] for more details).

When the $(2+1)$D SOS surface is constrained to stay above a hard wall (or floor), Bricmont, El-Mellouki and Fröhlich [12] showed in 1986 the appearance of the *entropic repulsion*: for large enough β, the floor pushes the SOS surface to diverge even though $\beta > \beta_{\mathrm{R}}$. More precisely, using Pirogov-Sinaï theory (see the review [47]), the authors of [12] showed that the SOS surface on an $L \times L$ box rises, amid the penalizing zero boundary, to an average height $H(L)$ satisfying $(1/C\beta)\log L \leqslant H(L) \leqslant (C/\beta)\log L$ for some absolute constant $C > 0$, in favor of freedom to create spikes downwards.

Entropic repulsion is one of the key features of the physics of random surfaces. This phenomenon has been rigorously analyzed mainly for some continuous-height variants of the SOS model in which the interaction potential $|\eta_x - \eta_y|$ is replaced by a *convex* potential $V(\eta_x - \eta_y)$; see e.g. [7–9, 19, 55, 56], see also [2] for a recent analysis of the wetting transition in the SOS model. As we will see below, entropic repulsion has a profound impact not only on the *equilibrium shape* of the surface but also on its *time evolution* under natural Markovian dynamics for the interface. The rigorous analysis of these dynamical effects of entropic repulsion will be the central focus of this work.

The dynamics we consider is the heat bath dynamics, or Gibbs sampler, for the equilibrium measure π_Λ, i.e. the discrete time Markov chain where at each step a site $x \in \Lambda$ is picked at random and the height η_x of the surface at x is replaced by a random variable $\eta'_x \in \{0,\dots,n^+\}$ distributed according to the conditional probability $\pi_\Lambda(\cdot \mid \eta_y,\, y \neq x)$. This defines a Markov chain with state space Ω_{Λ,n^+}, reversible with respect to π_Λ, commonly referred to as the Glauber dynamics. As explained below, our results apply equally well to other standard choices of reversible Markov chains, such as e.g. the Metropolis chain where only moves of the type $\eta'_x = \eta_x \pm 1$ are allowed.

The mixing time $T_{\textsc{mix}}$ is defined as the number of steps needed to reach approximate stationarity with respect to total variation distance, see Section 2 for definitions.

The main result of this paper is that the mixing of Glauber dynamics for the (2+1)D SOS is exponentially slow, due to the nature of the entropic repulsion effect.

Theorem 1. *For any sufficiently large inverse-temperature β there is some $c(\beta) > 0$ such that the following holds for all $L \in \mathbb{N}$. The mixing time $T_{\textsc{mix}}$ of the Glauber dynamics of the $(2+1)$D SOS model on $\Lambda = \{1,\dots,L\}^2$ with zero boundary conditions, floor at zero and ceiling at n^+ with $\log L \leqslant n^+ \leqslant L$ satisfies*

$$e^{cL} \leqslant T_{\textsc{mix}} \leqslant e^{(1/c)L}\,. \tag{1.2}$$

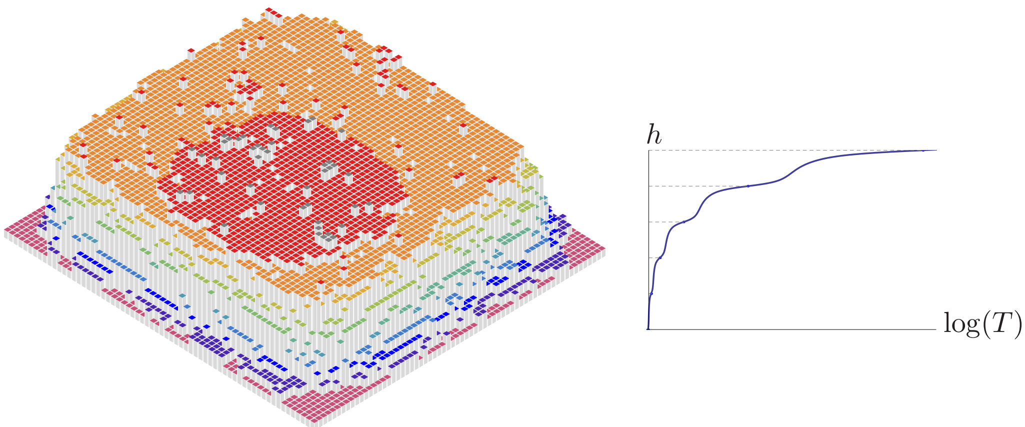

FIGURE 1. Illustration of the series of metastable states in the surface evolution. The dynamics waits time $e^{c\exp(4\beta h)}$ until the formation of a macroscopic droplet (marked in red) which eventually raises the average height from $h-1$ to h.

The exponentially large mixing time in (1.2) is in striking contrast with the rapid mixing displayed by Glauber dynamics of the $(1+1)$D SOS model [15, 44]. When $d=1$ it is known that the main driving effect is a mean-curvature motion which induces a diffusive relaxation to equilibrium, with T_{MIX} of order L^2 up to poly$(\log L)$ corrections. As we will see, in $(2+1)$D instead the main mechanism behind equilibration is a series of metastable transitions through an increasing series of effective energy barriers caused by the entropic repulsion. This is also in contrast with the behavior of related interface models with continuous heights as e.g. in [20, 24].

1.1. **Metastability and entropic repulsion.** Consider the evolution of an initially flat surface at height zero. We shall give a rough description of how it rises to the final height $H(L)$ through a series of metastable states indexed by $h \geqslant 0$. Roughly speaking the surface in state with label h is approximately flat at height h with rare up or downward spikes. Of course downward spikes cannot be longer than h because of the hard wall. If $h < H(L)$ then the surface has an advantage to rise to the next level $h+1$. This is due to the gain in entropy, measured by the possibility of having *downward* spikes of length $h+1$, beating the energy loss from the zero boundary conditions.

The mechanism for jumping to the next level should then be very similar to that occurring in the 2D Ising model at low temperature with a small external field opposite to the boundary conditions (see [53, 54]). Specifically, via a large deviation the surface at height h creates a large enough droplet of sites at height $h+1$ which afterwards expands to cover most of the available area. The energy/entropy balance of any such droplet is roughly[1] of order $\beta|\gamma| - e^{-4\beta(h+1)}A(\gamma)$ where $|\gamma|$ and $A(\gamma)$ are the boundary length and area respectively and the effective field $e^{-4\beta(h+1)}$ represents the probability of a $1\times 1\times (h+1)$ isolated downward spike. Simple considerations suggest then that the critical length of a droplet should be proportional to $e^{4\beta(h+1)}$. Finally the well established metastability theory for the 2D Ising model indicates that the activation time T_h for such a critical droplet should be exponential in the critical length[2] (i.e. a double exponential in h) as seen in Figure 1.

[1] Here we are neglecting finer results taking into account the surface tension and the associated Wulff theory; the basic conclusions of this reasoning are nevertheless still valid.

[2] At the early stages of the process when h is quite small the activation time has important corrections to this guess due to the many locations in the $L\times L$ box where the droplet can appear. However, as soon as h becomes of order $\log\log L$ these entropic corrections become negligible.

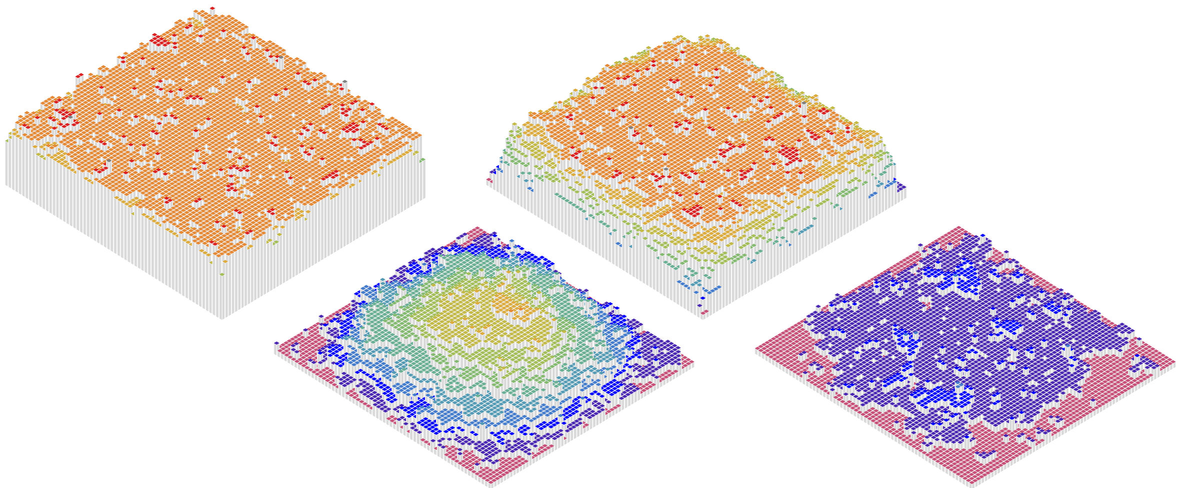

FIGURE 2. Glauber dynamics for SOS on a 64×64 square lattice at $\beta = 0.9$ from an initial state $\eta \equiv 10$. Surface gradually falls towards level $H = 1$. Snapshots at $t = 10$ (top left), $t = 100$, $t = 1000$, $t = 10000$ (bottom right) in cont. time.

Of course, in order to establish, even partially, the above picture and to prove the asymptotic of $\log(T_{\text{MIX}})$ as per (1.2) it is imperative to estimate the final equilibrium height of the surface $H(L)$ to within an *additive* $O(1)$. In Section 3 (Theorem 3.1), we improve the estimates of [12] to show that in fact the typical height of the surface at equilibrium is $H(L) + O(1)$, where

$$H(L) = \left\lfloor \frac{1}{4\beta} \log L \right\rfloor . \tag{1.3}$$

The aforementioned picture of the evolution of the SOS surface through a series of metastable states is quantified by the following result.

Theorem 2. *For any sufficiently large inverse-temperature β there is some $c(\beta) > 0$ such that the following holds. Let $(\eta(t))_{t\geqslant 0}$ be the Glauber dynamics for the SOS model on $\Lambda = \{1,\dots,L\}^2$ with zero boundary conditions, floor at zero and ceiling at n^+ with $\log L \leqslant n^+ \leqslant L$, started from the all-zero initial state. Fix $a \in (0,1)$ and let $\tau_a = \min\{t : \eta(t) \in \Omega_a\}$ where*

$$\Omega_a = \left\{\eta \in \Omega_{\Lambda,n^+} \ : \ \#\{x : \eta_x \geqslant aH(L)\} > \tfrac{9}{10}|\Lambda|\right\} . \tag{1.4}$$

Then $\lim_{L\to\infty} \pi_\Lambda(\Omega_a) = 1$ and yet

$$\lim_{L\to\infty} \mathbb{P}\left(e^{cL^a} \leqslant \tau_a \leqslant e^{(1/c)L^a}\right) = 1\,. \tag{1.5}$$

In fact, we prove this with the constant $\frac{9}{10}$ in (1.4) replaced by $1-\varepsilon(\beta)$ where $\lim_{\beta\to\infty}\varepsilon(\beta) = 0$. Moreover, the statement of the above theorem remains valid when $a = a(L) \to 1$ as long as the target level $h = aH(L)$ satisfies $h \leqslant H(L) - c$ for some sufficiently large $c(\beta) > 0$.

Remark. *A natural conjecture in light of Theorem 2 is that there exists a constant λ such that the distribution of $\tau_a \times e^{-\lambda L^a}$ converges as $L \to \infty$ to an exponential random variable.*

We wish to emphasize that, as will emerge from the proof, the exponential slowdown of equilibration is a coupled effect of entropic repulsion *and* of the rigidity of the interface. In particular, the following rough upper bound shows that the situation is very much different when the floor constraint is absent (yet the ceiling constraint remains unchanged):

Theorem 3. *Consider the (2+1)D SOS setting as in Theorem 1 with the exception that the surface heights belong to the* symmetric *interval $[-n^+, n^+]$. Then $T_{\text{MIX}} \leqslant \exp(o(L))$.*

Specifically, our proof gives the estimate $T_{\rm MIX} \leqslant \exp(L^{\frac12+o(1)})$. No effort was made to improve the exponent $\frac12$ as we would expect the true mixing behavior to be polynomial in L. We further expect that in the presence of a floor yet for $\beta < \beta_{\rm R}$ the mixing time will have a different scaling with the side-length L.

It is useful to compare our results with those of [17], where the Glauber dynamics for the $(2+1)$D SOS above a hard wall, at low temperature and in the presence of a weak attracting (towards the wall) external field was analyzed in details. There it was proved that certain critical values of the external field induce exponentially slow mixing while for all other values the dynamics is rapidly mixing. Although the slow mixing proved in [17] is similar to the one appearing in (1.2), the physical phenomenon behind it is very different. When an external field is present, a critical value of it results in two possible and roughly equally likely heights for the surface. In this case slow mixing arises because of the presence of a typical bottleneck in the phase space related to the bi-modal structure of the equilibrium distribution. In the setting of Theorems 1 and 2 instead, there is in general no bi-modal structure of the Gibbs measure and the slow mixing takes place because of a *multi-valley* structure of the effective energy landscape induced by the entropic repulsion which produces a whole family of bottlenecks.

1.2. **Methods.** We turn to a description of the main techniques involved in the proof of the main theorems. Our results can be naturally divided into three families: equilibrium estimates, lower bounds on equilibration times, and upper bounds on equilibration times.

Equilibrium Estimates. Our proof begins by deriving estimates for the equilibrium distribution which are crucial to the understanding of the dynamics (as discussed in Section 1.1) and of independent interest. Over most of the surface, the height is concentrated around $H(L)$ as defined in (1.3) with typical fluctuations of constant size. Achieving estimates with a precision level of an additive $O(1)$ turns out to be essential for establishing the order of the mixing time exponent: indeed, analogous estimates up to some additive $g(L)$ tending to ∞ with L would set off this exponent by a factor of $e^{O(g)}$.

The main techniques deployed for this part are a range of Peierls-type estimates for what we refer to as h-contours, defined as closed dual circuits with values at least h on the sites along their interior boundary and at most $h-1$ along their exterior boundary. In the simpler setting of no floor or ceiling (i.e., the sites are free to take all values in $\mathbb{Z}$ as their heights), the map S_γ which decreases all sites inside an h-contour γ by 1 is bijective and increases the Hamiltonian by $|\gamma|$, the length of the contour. Hence, the probability of a given h-contour in this setting is bounded by $\exp(-\beta|\gamma|)$. Iterating estimates of this form allows us to bound the deviations of the sites with the correct asymptotic in the setup of having no walls.

The presence of a floor renders this basic Peierls argument invalid since the map S_γ may leave sites in the interior with negative values. Rather than a technicality, this in fact lies at the heart of the entropic repulsion effect. We resort to estimating the probability that a given h-contour has a strictly positive interior, a quantity directly involving its area. By analyzing an isoperimetric tradeoff between the contour's area and perimeter we show that large contours above height $H(L)$ are unlikely, which in turn implies $O(1)$ typical fluctuations above this level. For a lower bound on the typical height of the surface we show that if too many sites are below $H(L)-k$ then the loss in energy due to raising the entire surface by 1 is more than compensated by the increased entropy from the freedom to create downward spikes reaching 0. Put together, these estimates guarantee that the height of most sites is within a constant of $H(L)$.

Equilibration times: lower bounds. Fix $h = aH(L)-1$ with $a \in (0,1)$ and consider the restricted ensemble obtained by conditioning the equilibrium measure on the event A that all h-contours have area smaller than δL^{2a}, for some small $\delta > 0$. Our equilibrium estimates imply that in this restricted ensemble

(i) each h-contour is actually very small (e.g. with area less than $\log(L)^2$), with very high probability;
(ii) the probability of the boundary of A is $O(\exp(-cL^a))$;
(iii) the probability of having a large density of heights at least $h+1 = aH(L)$ is $O(\exp(-cL^a))$.

In some sense (i), (ii) and (iii) above establish a bottleneck the Markov chain must pass through and thus provide the sought lower bound of $\exp(c(\beta)L^a)$ on the typical value of the hitting time τ_a in Theorem 2 when the initial state is the all zero configuration. In fact, the initially flat configuration can be replaced by monotonicity by the restricted ensemble described above. Then, in order for τ_a to be smaller than T, either the dynamics has gone through the boundary of A before T or the event described in (iii) occurred without leaving A. Either way an event with $O(\exp(-cL^a))$ probability occurred and the minimal time to see it must be proportional to the inverse of its probability.

Equilibration times: upper bounds. By the monotonicity of the system it is enough to consider the chain starting from the maximum and minimum configurations. The natural approach is to apply the well-known canonical paths method (see [21, 22, 35, 50] for various flavors of the method). As the cut-width of the cube is L^2, the most naïve application of this approach would give a bound of $\exp(O(L^2))$. A better bound can be shown by considering the problem with maximum height $n^+ = \log L$. In this case the cut-width is of order $L \log L$ yielding a mixing time upper bound of $\exp(O(L \log L))$. Since the height fluctuations are logarithmic, we can iterate this analysis using monotonicity and censoring to get a bound of $\exp(O(L \log L))$ for the original model with $n^+ = L$, vs. our lower bound of $\exp(cL)$. However, removing the $\log L$ factor that separates these exponents entails a significant amount of extra work.

The basic structure of the proof is to first establish a burn-in phase where we show that, starting from the maximal and minimal configurations, the process reaches a "good" set featuring small deviations from the equilibrium level $H(L)$. From there we establish a modified canonical paths estimate (Theorem 2.4), showing that it is enough to establish a reasonable probability of hitting the good set from any starting location together with a good canonical paths estimate restricted to this set. This new tool, which we believe is of interest on its own right, is described in detail in Section 2.3 and proved in a general context in Section 5.

Showing that the surface falls down from the ceiling (the maximum height) to $H(L)$, as depicted in Figure 2, ought to have been the easier part of the burn-in argument since high above the floor there is no entropic repulsion effect. Unfortunately a number of major technical challenges must be overcome.

First, the effect of the entropic repulsion is still apparent for the estimates we require when the surface is fairly close to $H(L)$. To overcome this we add a small external field to the model, thereby modifying the mixing time by a factor of at most $\exp(O(L))$ (which is large but still of the same order as our designated upper bound) and tilting the measure to remove these entropic repulsion effects. Second, while our main equilibrium estimates were proved using Peierls-type estimates, for the burn-in we require some of the cluster expansion machinery of [23] which we extend to the SOS framework. This involves a number of challenges including showing that the contours we consider do not interact significantly with the boundary conditions, a highly non-trivial fact. Implementing this scheme is the biggest challenge of the paper and we provide extensive notes for the reader in these sections to explain the rather technical proofs.

Finally, the fact that the surface rises from the floor (the all zero initial condition) to the vicinity of the equilibrium height $H(L)$ in time $\exp(O(L))$ is proved via an unusual inductive scheme. Unlike other multi-scale inductive schemes, somewhat surprisingly the one used here does not incur any penalizing factor on the upper bound. We first prove weaker bounds on the mixing time and use these estimates to show that a smaller box of side-length $L/\log L$ mixes by time $\exp(O(L))$. By monotonicity, we can use this to bound the distance from the equilibrium height of the surface in the original box by $H(L) - H(L/\log L)$. By using this height estimate along with our canonical paths result we get improved bounds on the mixing time. This in turn allows us to take larger sub-boxes and iteratively achieve better and better estimates on the distance to $H(L)$. After sufficiently many iterations we show that the surface reaches height $H(L)-O(1)$ in time $\exp(O(L))$ and thereafter the canonical paths estimate completes the proof.

1.3. **Related open problems.**

Tilted walls. An interesting and to our knowledge widely open problem concerns the SOS model with a non-horizontal hard wall, i.e. when the constraint $\eta_x \geqslant 0$ is replaced by $\eta_x \geqslant \phi^{\mathbf{n}}_x$, where $\phi^{\mathbf{n}}_x$ denotes the discrete approximation of the plane orthogonal to the unit vector $\mathbf{n}$, and $\mathbf{n}$ is assumed to have all components different from zero. The equilibrium fluctuations for $\beta = +\infty$ can be analyzed via their representation through dimer coverings [36] and the variance of the surface height in the middle of the box can be shown to be $O(\log L)$; see [14, Section 5] for a proof. Moreover, at $\beta = +\infty$, as far as the dynamics is concerned, it has been proved [15] that the mixing time is of order L^2 up to polylog(L) corrections and that the relaxation process is driven by mean curvature motion. The case $\beta < +\infty$, however, remains open both for equilibrium fluctuations and for mixing time bounds.

Mixing time for Ising model. In view of the natural connection with the Ising model, the study of Glauber dynamics for the SOS can also shed some light on a, still open, central problem in the theory of stochastic Ising models: its mixing time under an *all-plus* boundary in the phase coexistence region. The long-standing conjecture is that the mixing time of Glauber dynamics for the Ising model on a box of side-length L with all-plus boundary should be at most polynomial in L at *any temperature*. More precisely, the convergence to equilibrium should be driven by a mean-curvature motion of the interface of the minus droplet in accordance with Lifshitz's law [40]. For instance, the mixing time of Glauber dynamics for Ising on an $L \times L$ square lattice is conjectured [26] to be of order L^2 in continuous time. This was confirmed at zero temperature [18, 27, 38] and near-zero temperatures [14], yet the best known upper bound for finite $\beta > \beta_c$ remains quite far, a quasi-polynomial bound of $L^{O(\log L)}$ due to [41]. The understanding of 3D Ising is far more limited: while at zero temperature bounds of $L^{2+o(1)}$ were recently proven in [14], no sub-exponential mixing bounds are known at any finite $\beta > \beta_c$.

2. Definitions and tools

2.1. **Glauber dynamics for Solid-On-Solid.** Let $\sqcup$ and $\sqcap$ denote the minimal and maximal configurations in Ω_{Λ,n^+}, i.e. $\sqcup_x = 0$ and $\sqcap_x = n^+$ for every $x \in \Lambda$. Given a finite connected subset $\Lambda \subset \mathbb{Z}^2$, let $\partial\Lambda$ denote its external boundary, i.e. the set of sites in Λ^c which are at distance 1 from Λ. To extend the SOS definition to arbitrary boundary conditions (b.c.) given by $\xi : \mathbb{Z}^2 \to \mathbb{Z}$, define the SOS Hamiltonian with b.c. ξ to be

$$\mathcal{H}^{\xi}_{\Lambda}(\eta) := \frac{1}{2} \sum_{\substack{x,y\in\Lambda \\ |x-y|=1}} |\eta_x - \eta_y| + \sum_{\substack{x\in\Lambda,\ y\in\partial\Lambda \\ |x-y|=1}} |\eta_x - \xi_y|. \tag{2.1}$$

Given $\beta > 0$ and n^+, the Gibbs measure π_Λ^ξ on Ω_{Λ,n^+} with b.c. ξ is defined as

$$\pi_\Lambda^\xi(\eta) = \frac{1}{Z_\Lambda^\xi} \exp\Big[-\beta\, \mathcal{H}_\Lambda^\xi(\eta)\Big]. \tag{2.2}$$

Notation 2.1. *In the sequel when the b.c. $\xi \equiv n \in \mathbb{Z}$ we will use the abbreviated form π_Λ^n. We will occasionally drop the subscript Λ and superscript ξ from the notation of π_Λ^ξ when there is no risk of confusion. Moreover we will need to address the following variants of π_Λ^ξ:*

(i) the measure $\hat\pi_\Lambda^n$ of SOS without walls (no floor and no ceiling) and with b.c. at height n;
(ii) the measure Π_Λ^ξ corresponding to π_Λ^ξ with $n^+ = +\infty$ (no ceiling);
(iii) starting from Section 6 the measures $\pi_\Lambda^{\xi,f}$ (and its analog $\Pi_\Lambda^{\xi,f}$ with no ceiling) corresponding to the SOS Hamiltonian with an additional external field of the form $\frac{1}{L}\sum_{y\in\Lambda} f(\eta_y)$ with $|f|_\infty = O(e^{-c\beta})$ for some fixed constant c (see e.g. (6.1)).

The dynamics under consideration is a discrete-time Markov chain $(\eta(t))_{t=0,1,\ldots}$, defined as follows. To construct $\eta(t+1)$ given $\eta(t)$,

- pick a site $x \in \Lambda$ uniformly at random;
- sample a new value for $\eta_x(t+1)$ from the equilibrium measure π_Λ^ξ conditioned on the current heights at the neighboring sites, that is, $\eta(t+1) \sim \pi_\Lambda^\xi(\eta \in \cdot \mid \eta_y = \eta_y(t)\ \forall y \neq x)$.

The law of the process with initial condition ζ is denoted by $\mathbb{P}^\zeta$, the configuration at time t is $\eta^\zeta(t)$ and its law is μ_t^ζ. When there is no need to emphasize the initial condition we simply write $\eta(t)$ for the configuration at time t. It is well-known that this Markov chain is reversible w.r.t. the invariant measure π_Λ^ξ.

The mixing time $T_{\textsc{mix}}$ is defined to be the time the process takes to converge to equilibrium in total variation distance, that is,

$$T_{\textsc{mix}} = \inf\left\{ t > 0 : \max_{\eta\in\Omega_{\Lambda,n^+}} \|\mu_t^\eta - \pi_\Lambda^\xi\| \leqslant \frac{1}{2e}\right\} \tag{2.3}$$

where $\|\mu - \nu\|$ denotes the total variation distance between two measures μ, ν. It is well-known (e.g., [39, Section 4.5]) that the total variation distance from equilibrium decays exponentially with rate $T_{\textsc{mix}}$, namely

$$\max_{\eta\in\Omega_{\Lambda,n^+}} \|\mu_t^\eta - \pi_\Lambda^\xi\| \leqslant e^{-\lfloor t/T_{\textsc{mix}}\rfloor}\,. \tag{2.4}$$

The relaxation time $T_{\textsc{rel}}$ is the inverse of the spectral gap of the transition kernel of the chain. The spectral gap, denoted by $\mathtt{gap}$, has the following variational characterization:

$$\mathtt{gap} = \inf \frac{\pi_\Lambda^\xi(f(I-P)f)}{\mathrm{Var}_{\pi_\Lambda^\xi}(f)}\,, \tag{2.5}$$

where P is the transition kernel of the chain, I is the identity matrix and the infimum is over all non-constant functions f. The following standard inequality (see, e.g., [39, Section 12.2] and [49]) relates the mixing time and the relaxation time:

$$T_{\textsc{rel}} - 1 \leqslant T_{\textsc{mix}} \leqslant T_{\textsc{rel}} \log(2e/\pi_{\min}) \tag{2.6}$$

with $\pi_{\min} := \min_{\eta\in\Omega_{\Lambda,n^+}} \pi_\Lambda^\xi(\eta)$. By definition, in the SOS model $|\Omega_{\Lambda,n^+}| = (n^+ + 1)^{|\Lambda|}$ and $\pi_{\min} \geqslant \exp(-4\beta|\Lambda|n^+)/|\Omega_{\Lambda,n^+}|$, thus for large enough n^+

$$T_{\textsc{rel}} - 1 \leqslant T_{\textsc{mix}} \leqslant 5\beta|\Lambda| n^+\, T_{\textsc{rel}}\,. \tag{2.7}$$

From now on we refer to the Markov chain defined above as the Glauber dynamics. One can use standard comparison estimates to obtain equivalent versions of our main results for other standard choices of Markov chains that are reversible w.r.t. the SOS Gibbs measures, such as e.g. the Metropolis chain with ± 1 updates. Indeed, since the heights are confined within an interval of size $O(L)$ it is not hard to see that the the ratio between the different mixing times is at most polynomial in L. We refer to e.g. [14, Section 6] for a detailed argument in this direction.

2.2. **Monotonicity.** Our dynamics is monotone (or attractive) in the following sense. One equips the configuration space with the natural partial order such that $\sigma \leqslant \eta$ if $\sigma_x \leqslant \eta_x$ for every $x \in \Lambda$. It is possible to couple on the same probability space the evolutions corresponding to every possible initial condition ζ and boundary condition ξ in such a way that if $\xi \leqslant \xi'$ and $\zeta \leqslant \zeta'$ then $\eta^\zeta(t,\xi) \leqslant \eta^{\zeta'}(t,\xi')$ for every t. Here, we indicated explicitly the dependence on the boundary conditions but we will not do so in the following. The law of the *global monotone coupling* is denoted $\mathbb{P}$.

A first consequence of monotonicity is that the FKG inequalities [28] hold: if f and g are two increasing (w.r.t. the above partial ordering) functions, then $\pi^\xi_\Lambda(fg) \geqslant \pi^\xi_\Lambda(f)\pi^\xi_\Lambda(g)$ and the same holds for the measure $\hat\pi^\xi_\Lambda$ without the floor/ceiling.

Monotonicity also implies the following standard fact (cf. e.g. the proof of [45, Eq. (2.10)]): for every initial condition η and boundary condition ξ,

$$\|\mu_t^\eta - \pi^\xi_\Lambda\| \leqslant 2n^+|\Lambda| \max(\|\mu_t^\sqcup - \pi^\xi_\Lambda\|, \|\mu_t^\sqcap - \pi^\xi_\Lambda\|). \tag{2.8}$$

Another consequence of monotonicity is the so-called Peres-Winkler censoring inequality. Take integers $0 = t_0 < t_1 < \cdots < t_k = T$, a sequence of $V_i \subset \Lambda$ and $0 \leqslant a_i \leqslant b_i \leqslant n^+, i \leqslant k$. Consider the following modified dynamics $(\tilde\eta(t))_{0 \leqslant t \leqslant T}$. To construct $\tilde\eta(t+1)$ given $\tilde\eta(t)$,

- pick a site $x \in \Lambda$ uniformly at random;
- at time t with $t_{i-1} < t \leqslant t_i$ do as follows:
  - if $x \notin V_i$ or if $x \in V_i$ and $\tilde\eta_x(t) \notin \{a_i, \dots, b_i\}$ then do nothing;
  - if $x \in V_i$ and $a_i \leqslant \tilde\eta_x(t) \leqslant b_i$ then replace its value with a new value $\tilde\eta_x(t+1)$ in $\{a_i, \dots, b_i\}$ with probability proportional to the stationary measure conditioned on the value of the neighboring columns,

$$\tilde\eta(t+1) \sim \pi^\xi_\Lambda\big(\eta \in \cdot \mid \eta_x \in \{a_i, \dots, b_i\}\ ,\ \eta_y = \tilde\eta_y(t)\ \forall y \neq x\big)\,.$$

Call $\tilde\mu_t^\nu$ the law at time t when the initial distribution is ν. The following then holds:

Theorem 2.2 (special case of [46, Theorem 1.1])**.** *If the initial distribution ν is such that $\nu(\eta)/\pi^\xi_\Lambda(\eta)$ is an increasing (resp. decreasing) function, then $\tilde\mu_t^\nu(\eta)/\pi^\xi_\Lambda(\eta)$ is also increasing (resp. decreasing) for $t \leqslant T$ and $\mu_t^\nu \preceq \tilde\mu_t^\nu$ (resp. $\tilde\mu_t^\nu \preceq \mu_t^\nu$). In addition,*

$$\|\mu_t^\nu - \pi^\xi_\Lambda\| \leqslant \|\tilde\mu_t^\nu - \pi^\xi_\Lambda\|\,. \tag{2.9}$$

2.3. **An improved path argument.** Geometric techniques can prove very effective in getting upper bounds on the relaxation time and therefore on the mixing time of a Markov chain [21, 22, 35, 50] (see also [39, Section 13.5]). Let us recall the basic principle.

Let $(X(t))_{t=0,1,\dots}$ be a discrete-time reversible Markov chain on a finite state space Ω, with invariant measure π. For $a, b \in \Omega$ such that the one-step transition probability $p(a,b)$ from a to b is non-zero, set $Q(a,b) = \pi(a)p(a,b) = Q(b,a)$. For each couple $(c,d) \in \Omega^2$ fix a path $\gamma(c,d) = (x_1, \dots, x_n)$ in Ω with $x_1 = c, x_n = d$ and $p(x_i, x_{i+1}) \neq 0$ and let $|\gamma(c,d)| := n$. Then the relaxation time of the Markov chain is bounded as

$$T_{\text{REL}} \leqslant \max_{(a,b):Q(a,b)\neq 0} \frac{1}{Q(a,b)} \sum_{\substack{\eta,\eta'\in\Omega:\\(a,b)\in\gamma(\eta,\eta')}} |\gamma(\eta,\eta')|\pi(\eta)\pi(\eta'). \tag{2.10}$$

Here, $(a,b)\in\gamma(\eta,\eta')$ means that if $\gamma(\eta,\eta')=(x_1,\ldots,x_n)$ then there exists i such that $a=x_i, b=x_{i+1}$. The proof is simply an application of the Cauchy-Schwarz inequality; see e.g. [49].

An application of this principle gives the following:

Proposition 2.3. *For the SOS dynamics in the $\Lambda=\{1,\ldots,L\}\times\{1,\ldots,m\}, m\leqslant L$, with floor at height zero, ceiling at n^+ and b.c. ξ, one has for some $c=c(\beta)$*

$$T_{\text{REL}} \leqslant cL^2m^2n^+\,\exp(7\beta m n^+) \tag{2.11}$$

and, thanks to (2.7), $T_{\text{MIX}}=\exp(O(\beta L n^+))$ if $L=m$.

That (2.11) easily follows from (2.10) was observed in [43] in the case of the Glauber dynamics of the Ising model (in this case one refers to the paths $\gamma(\eta,\eta')$ as "canonical paths"). For SOS the proof is very similar and is given for completeness in Section 5.1.

However this upper bound is too rough for our purposes since we have $n^+\geqslant \log L$ while we wish to get a mixing time upper bound which is exponential in L. Therefore a significant part of the present work is devoted to getting rid of the non-physical factor n^+ in the argument of the exponential in the r.h.s. of (2.11). Although this task may appear to be mainly of technical nature it actually requires a much deeper understanding of the actual behavior of the dynamics compared to that provided by canonical paths, and the support of new ideas.

One of the key ingredients we use is the following improved version of (2.10), which we believe can be interesting in a more general context:

Theorem 2.4. *Let $G\subset\Omega$ and assume that, for some $T>0$ and for every initial condition x, $\mathbb{P}^x(X(T)\in G)\geqslant\alpha$ with $\mathbb{P}^x$ denoting the law of the chain starting at x. Assume further that for every η,η' in G there exists a path $\widetilde{\gamma}(\eta,\eta')$ as above which stays in G and let*

$$W(G) := \max_{\substack{a,b\in G\\Q(a,b)\neq 0}} \frac{1}{Q(a,b)} \sum_{\substack{\eta,\eta'\in G:\\(a,b)\in\widetilde{\gamma}(\eta,\eta')}} |\widetilde{\gamma}(\eta,\eta')|\pi(\eta)\pi(\eta'). \tag{2.12}$$

Then,

$$\mathtt{gap}^{-1} \leqslant \frac{6}{\alpha}\left(\frac{T^2}{p_{\min}}+\frac{W(G)}{\alpha}\right) \tag{2.13}$$

with $p_{\min}:=\min\{p(\sigma,\sigma')>0:\sigma,\sigma'\in\Omega\}$

This is clearly an improvement provided that α is bounded away from zero, that $W(G)\ll W(\Omega)$ and that T is not too large (in simple words, we need that with non-zero probability the chain enters "quickly" the good set G where canonical paths work well).

In our SOS application, roughly speaking, we will choose G to be the set of configurations such that $|\Lambda_L|^{-1}\sum_{x\in\Lambda_L}|\eta_x-H(L)|$ is upper bounded by a constant. We will see that, irrespective of the starting configuration, at time $T=\exp(O(L))$ the dynamics is in G with probability at least $\frac{1}{2}$. On the other hand, a minor modification of Proposition 2.3 will give $W(G)=\exp(O(\beta L))$. Then, Theorem 2.4 allows us to improve the mixing time upper bound to $T_{\text{MIX}}=\exp(O(\beta L))$.

3. Equilibrium results

Theorem 3.1. *Let $\Lambda \subset \mathbb{Z}^2$ be a box of side-length L and let $\beta \geqslant 1$. Set $H = \lfloor \frac{1}{4\beta} \log L \rfloor$. There exist some absolute constants $C, K > 0$ (with K integer) such that for any integer $k \geqslant K$,*

$$\pi_\Lambda^0\left(\#\{v : \eta_v \leqslant H - k\} > e^{-2\beta k} L^2\right) \leqslant \exp\left(-e^{\beta k} L\right), \tag{3.1}$$

$$\pi_\Lambda^0\left(\#\{v : \eta_v \geqslant H + k\} > e^{-2\beta k} L^2\right) \leqslant \exp\left(-C e^{-2\beta k} L \big(1 \wedge e^{-2\beta k} L \log^{-8} L\big)\right). \tag{3.2}$$

(Notice that the bound on downward fluctuations improves with the size of the deviation whereas the bound on upward fluctuations deteriorates with the distance.)

Recall that π_Λ^0 has a floor at 0 and a ceiling at height $\log L \leqslant n^+ \leqslant L$ (together with zero boundary conditions). It will be convenient throughout this section to work in the setting of a floor at 0 but no ceiling, where the corresponding measure Π_Λ^0 is asymptotically equal to π_Λ^0:

Lemma 3.2. *There is an absolute constant $c > 0$ such that for any $\beta \geqslant 1$ and any subset of configurations $A \subseteq \{0, \ldots, n^+\}^\Lambda$,*

$$\Pi_\Lambda^0(A) \leqslant \pi_\Lambda^0(A) \leqslant \left(1 + cL^2 e^{-2\beta n^+}\right) \Pi_\Lambda^0(A).$$

The above lemma, which will be proved further on in this section, entitles us to derive results on π_Λ^0 from Π_Λ^0 at an asymptotically negligible cost.

The following notion of a *contour* and that of an *h-contour*, a level line at height h, play a crucial role in our proofs.

Definition 3.3. *We let $\mathbb{Z}^{2^*}$ be the dual lattice of $\mathbb{Z}^2$ and we call a bond any segment joining two neighboring sites in $\mathbb{Z}^{2^*}$. Two sites x, y in $\mathbb{Z}^2$ are said to be* separated by a bond e *if their distance (in $\mathbb{R}^2$) from e is $\frac{1}{2}$. A pair of orthogonal bonds which meet in a site $x^* \in \mathbb{Z}^{2^*}$ is said to be a* linked pair of bonds *if both bonds are on the same side of the forty-five degrees line across x^*. A* geometric contour *(for short a contour in the sequel) is a sequence $e_0, \ldots, e_n$ of bonds such that:*

1. *$e_i \neq e_j$ for $i \neq j$, except for $i = 0$ and $j = n$ where $e_0 = e_n$.*
2. *for every i, e_i and e_{i+1} have a common vertex in $\mathbb{Z}^{2^*}$*
3. *if $e_i, e_{i+1}, e_j, e_{j+1}$ intersect at some $x^* \in \mathbb{Z}^{2^*}$, then e_i, e_{i+1} and e_j, e_{j+1} are linked pairs of bonds.*

We denote the length of a contour γ by $|\gamma|$, its interior (the sites in $\mathbb{Z}^2$ it surrounds) by Λ_γ and its interior area (the number of such sites) by $|\Lambda_\gamma|$. Moreover we let Δ_γ be the set of sites in $\mathbb{Z}^2$ such that either their distance (in $\mathbb{R}^2$) from γ is $\frac{1}{2}$, or their distance from the set of vertices in $\mathbb{Z}^{2^*}$ where two non-linked bonds of γ meet equals $1/\sqrt{2}$. Finally we let $\Delta_\gamma^+ = \Delta_\gamma \cap \Lambda_\gamma$ and $\Delta_\gamma^- = \Delta_\gamma \setminus \Delta_\gamma^+$.

Definition 3.4. *Given a contour γ we say that γ is an h-*contour *for the configuration η if*

$$\eta\restriction_{\Delta_\gamma^-} \leqslant h - 1, \quad \eta\restriction_{\Delta_\gamma^+} \geqslant h.$$

We will say that γ is a contour for the configuration η if there exists h such that γ is a h-contour for η. Finally $\mathscr{C}_{\gamma,h}$ will denote the event that γ is an h-contour.

To illustrate the above definitions with a simple example, consider the elementary contour given by the square of side 1 surrounding a site $x \in \mathbb{Z}^2$. In this case, γ is an h-contour iff $\eta_x \geqslant h$ and $\eta_y \leqslant h - 1$ for all $y \in \{x \pm e_1, x \pm e_2, x + e_1 + e_2, x - e_1 - e_2\}$. In general, Δ_γ^+ (resp. Δ_γ^-) is the set of $x \in \Lambda_\gamma$ (resp. $x \in \Lambda_\gamma^c$) either at distance 1 from Λ_γ^c (resp. Λ_γ) or at distance $\sqrt{2}$ from a vertex $y \in \Lambda_\gamma^c$ (resp. $y \in \Lambda_\gamma$) in the south-west or north-east direction.

Remark 3.5. *As the reader may have noticed the definition of an h-contour is asymmetric in the sense that we require the minimal height of the surface at the inner boundary of γ, Δ_γ^+, to be larger than the maximum height at the external boundary. In a sense this definition covers* upward fluctuations *of the surface. Of course one could provide the reverse definition covering* downward fluctuations. *In the sequel the latter is not really needed thanks to monotonicity and symmetry arguments. We also observe that, contrary to what happens in e.g. Ising models, a geometric contour γ could be at the same time a h-contour and a h'-contour with $h \neq h'$. More generally two geometric contours γ, γ' could be contours for the same surface with different height parameters even if $\gamma \cap \gamma' \neq \emptyset$ (but one of them must be contained in the other).*

The following estimates play a key role in the proof of Theorem 3.1.

Proposition 3.6. *There exists an absolute constant $C_0 > 0$ such that for all $\beta \geqslant 1$ and $h \geqslant 1$,*

$$\pi_\Lambda^0\left(\mathscr{C}_{\gamma,h}\right) \leqslant \exp\left(-\beta|\gamma| + C_0|\Lambda_\gamma|e^{-4\beta h}\right). \tag{3.3}$$

Moreover, for any family of h-contours $\{(\gamma_s, h_s)\}_{s\in\mathscr{S}}$ such that for all $i \geqslant 1$

$$\bigcup_{\substack{s\in\mathscr{S}\\ h_s=i+1}} \Lambda_{\gamma_s} \subseteq \bigcup_{\substack{s\in\mathscr{S}\\ h_s=i}} \Lambda_{\gamma_s}$$

and $\Lambda_{\gamma_s} \cap \Lambda_{\gamma_{s'}} = \emptyset$ when $h_s = h_{s'}$, $s \neq s'$,we have

$$\pi_\Lambda^0\bigg(\bigcap_{s\in\mathscr{S}} \mathscr{C}_{\gamma_s,h_s}\bigg) \leqslant \exp\bigg(\sum_{s\in\mathscr{S}}\Big(-\beta|\gamma_s| + C_0|\Lambda_{\gamma_s}|e^{-4\beta h_s}\Big)\bigg). \tag{3.4}$$

As a step towards the proof of the above proposition, we consider the setting of no floor and no ceiling, where the picture is simpler as there is no entropic repulsion.

Lemma 3.7. *For any h-contour γ in any domain Λ with any boundary condition ξ we have*

$$\hat\pi_\Lambda^\xi\left(\mathscr{C}_{\gamma,h}\right) \leqslant \exp(-\beta|\gamma|).$$

Moreover, if $h' < h$ and γ, γ' are contours with $\Lambda_\gamma \subseteq \Lambda_{\gamma'}$ then

$$\hat\pi_\Lambda^\xi\left(\mathscr{C}_{\gamma,h} \mid \mathscr{C}_{\gamma',h'}\right) \leqslant \exp(-\beta|\gamma|). \tag{3.5}$$

Proof. Define the map $T = T_\gamma : \mathbb{Z}^\Lambda \to \mathbb{Z}^\Lambda$ by

$$(T\eta)_v = \begin{cases} \eta_v - 1 & v \in \Lambda_\gamma, \\ \eta_v & \text{otherwise.} \end{cases} \tag{3.6}$$

If η has an h-contour at γ then the difference along every edge in $\mathbb{Z}^2$ crossing γ decreases by 1 so $\hat\pi_\Lambda^\xi(T\eta) = e^{\beta|\gamma|}\hat\pi_\Lambda^\xi(\eta)$. Since T is a bijection it follows that

$$\sum_{\mathscr{C}_{\gamma,h}} \hat\pi_\Lambda^\xi(\eta) = e^{-\beta|\gamma|} \sum_{T^{-1}(\mathscr{C}_{\gamma,h})} \hat\pi_\Lambda^\xi(T\eta) \leqslant e^{-\beta|\gamma|}.$$

Equation (3.5) follows from the same argument by noting that if $\eta \in \mathscr{C}_{\gamma,h} \cap \mathscr{C}_{\gamma',h'}$ then $T_\gamma\eta$ remains in $\mathscr{C}_{\gamma',h'}$. This completes the proof. ∎

Remark 3.8. *In the context of considering the interior of an h-contour γ for possibly nested contours (such as the ones featured in Eq. (3.5)), a useful observation is that*

$$\pi_\Lambda^0\left(\eta\restriction_{\Lambda_\gamma} \in \cdot \mid \mathscr{C}_{\gamma,h}\right) = \pi_{\Lambda_\gamma}^\xi\left(\cdot \mid \eta\restriction_{\Delta_\gamma^+} \geqslant h\right)$$

for any boundary condition ξ that is at most h all along Δ_γ^-. This follows from the fact that conditioning on any fixed $\xi \leqslant h$ would contribute an equal pre-factor to all configurations thanks to having $\eta\restriction_{\Delta_\gamma^+} \geqslant h$, and as this includes all ξ's with $\eta\restriction_{\Delta_\gamma^-} \leqslant h-1$ this further includes $\mathscr{C}_{\gamma,h}$. Moreover, the same holds when conditioning on $\mathscr{C}_{\gamma,h} \cap E$ (instead of just $\mathscr{C}_{\gamma,h}$) for an arbitrary event E which is only a function of the configuration on $(\Lambda_\gamma)^c$.

(Note that the above remark similarly applies to Π and $\hat\pi$ by the same argument.)

A Peierls-type argument will transform the above lemma into the following bound on upward (downward) fluctuations in the no floor, no ceiling setting.

Proposition 3.9. *There exists an absolute constant $c > 0$ such that for any $\beta \geqslant 1$, domain Λ, site $v \in \Lambda$ and height $h \geqslant 0$,*

$$\tfrac{1}{2} e^{-4\beta h} \leqslant \hat\pi_\Lambda^0\left(\eta_v \geqslant h\right) \leqslant c e^{-4\beta h}\,.$$

Proof. Define the map $S : \mathbb{Z}^\Lambda \to \mathbb{Z}^\Lambda$ by $(S\eta)_u = \eta_u$ for $u \neq v$ and

$$(S\eta)_v = \begin{cases} \eta_v + h & \eta_v \geqslant 0\,, \\ \eta_v - h & \eta_v < 0\,. \end{cases}$$

Observe that $|(S\eta)_v| \geqslant h$ and that since S changes the Hamiltonian by at most $4h$,

$$\hat\pi_\Lambda^0\left(S\eta\right) \geqslant e^{-4\beta h}\hat\pi_\Lambda^0\left(\eta\right)\,.$$

Moreover, as S is injective, summing over η we have that

$$\hat\pi_\Lambda^0\left(|\eta_v| \geqslant h\right) = \sum_{\eta\in\mathbb{Z}^\Lambda} \hat\pi_\Lambda^0\left(S\eta\right) \geqslant e^{-4\beta h} \sum_{\eta\in\mathbb{Z}^\Lambda} \hat\pi_\Lambda^0\left(\eta\right) = e^{-4\beta h}.$$

Since by symmetry $\hat\pi_\Lambda^0\left(\eta_v \geqslant h\right) = \hat\pi_\Lambda^0\left(\eta_v \leqslant -h\right)$ the lower bound follows.

To get the upper bound define a set of nested contours surrounding v as

$$\mathcal{A}(h,v) = \left\{(\gamma_1,\ldots,\gamma_h) : v \in \Lambda_{\gamma_h} \text{ and } \Lambda_{\gamma_{i+1}} \subseteq \Lambda_{\gamma_i} \text{ for all } 1 \leqslant i \leqslant h-1\right\}$$

and observe that, if η is such that $\eta_v \geqslant h$, then necessarily there exists $(\gamma_1,\ldots,\gamma_h) \in \mathcal{A}(h,v)$ such that $\eta \in \bigcap_{1\leqslant i\leqslant h} \mathscr{C}_{\gamma_i,i}$.

Applying Lemma 3.7 iteratively (while bearing Remark 3.8 in mind) we now obtain that for every $(\gamma_1,\ldots,\gamma_h) \in \mathcal{A}(h,v)$,

$$\hat\pi_\Lambda^\xi\left(\textstyle\bigcap_{1\leqslant i\leqslant h} \mathscr{C}_{\gamma_i,i}\right) \leqslant e^{-\beta\sum_{i=1}^h |\gamma_i|}\,. \tag{3.7}$$

Simple counting gives that the number of contours of length n starting from a vertex is at most R_n, the number of self avoiding walks of length n. If such a path surrounds v then it must cross the horizontal line containing v to its right within distance n so the number of γ with $|\gamma| = n$ and $v \in \Lambda_\gamma$ is at most nR_n (with room to spare). Hence

$$\sum_{\substack{\gamma\,:\,v\in\Lambda_\gamma\\ |\Lambda_\gamma|>2}} e^{-\beta|\gamma|+6\beta} \leqslant \sum_{n=8}^\infty n R_n e^{-\beta n+6\beta}\,,$$

which is uniformly bounded in β for any $\beta \geqslant 1$ since the connective constant $\mu_2 = \lim_{n\to\infty} R_n^{1/n}$ is known to satisfy $\mu_2 < 2.68 < e$. Hence, for some large enough M, independent of β,

$$\sum_{\substack{\gamma\,:\,v\in\Lambda_\gamma\\ |\Lambda_\gamma|>M}} e^{-\beta|\gamma|} \leqslant e^{-6\beta}\,. \tag{3.8}$$

Now define a collection of nested contours of area at least 2 and at most M as

$$\mathcal{A}_M(h,v) = \{(\gamma_1,\dots,\gamma_h) \in \mathcal{A}(h,v) : 2 \leqslant |\Lambda_{\gamma_i}| \leqslant M \text{ for all } 1 \leqslant i \leqslant h\}\ .$$

We note that

$$|\mathcal{A}_M(h,v)| \leqslant |\mathcal{A}_M(1,v)|^{M-1}\binom{h+M-2}{M-2} \leqslant |\mathcal{A}_M(1,v)|^{M-1}(h+M)^{M-2} \tag{3.9}$$

since, examining the way $|\Lambda_{\gamma_i}|$ decreases, there are at most $M-2$ transitions of $|\Lambda_{\gamma_i}| < |\Lambda_{\gamma_{i-1}}|$ and in each case the number of possible γ_i is at most $|\mathcal{A}_M(1,v)|$ with much room to spare.

For any $(\gamma_1,\dots,\gamma_h) \in \mathcal{A}(h,v)$ we can find $0 \leqslant k \leqslant l \leqslant h$ such that $|\Lambda_{\gamma_i}| > M$ for $1 \leqslant i \leqslant k$, that $(\gamma_{k+1},\dots,\gamma_l) \in \mathcal{A}_M(l-k,v)$ and $|\Lambda_{\gamma_i}| = 1$ for $l < i \leqslant h$. Then

$$\begin{aligned}
\sum_{(\gamma_1,\dots,\gamma_h)\in\mathcal{A}(h,v)} e^{-\beta\sum_{i=1}^h |\gamma_i|} &= \sum_{0\leqslant k\leqslant l\leqslant h}\ \sum_{\substack{(\gamma_1,\dots,\gamma_k)\\ |\Lambda_{\gamma_k}|>M}}\ \sum_{(\gamma_{k+1},\dots,\gamma_l)\in\mathcal{A}_M(l-k,v)} e^{-\beta\sum_{i=1}^l |\gamma_i| - 4\beta(h-l)} \\
&\leqslant \sum_{0\leqslant k\leqslant l\leqslant h} |\mathcal{A}_M(l-k,v)| e^{-6\beta l - 4\beta(h-l)} \\
&\leqslant e^{-4\beta h} \sum_{0\leqslant k\leqslant l\leqslant h} |\mathcal{A}_M(1,v)|^{M-1}(l-k+M)^{M-2} e^{-2\beta l} \\
&\leqslant c e^{-4\beta h}
\end{aligned}$$

where the first equality holds since $|\gamma_i| = 4$ when $|\Lambda_{\gamma_i}| = 1$, the inequality in the second line is by equation (3.8) and the fact that every contour with $|\Lambda_\gamma| \geqslant 2$ has $|\gamma| \geqslant 6$, and where the transition in the third line is by (3.9). Combining with equation (3.7) completes the proof. □

Proposition 3.9 allows us to readily infer Lemma 3.2:

Proof of Lemma 3.2. One has

$$\pi_\Lambda^0(A) = \Pi_\Lambda^0(A)\frac{\Xi_\Lambda^0}{Z_\Lambda^0}$$

where Ξ_Λ^0 denotes the partition function corresponding to the measure Π_Λ^0. The fact that $\Pi_\Lambda^0(A) \leqslant \pi_\Lambda^0(A)$ follows immediately from $\Xi_\Lambda^0 \geqslant Z_\Lambda^0$. To show that $\pi_\Lambda^0(A) \leqslant \left(1 + cL^2e^{-2\beta n^+}\right)\Pi_\Lambda^0(A)$, observe that

$$\frac{Z_\Lambda^0}{\Xi_\Lambda^0} = \Pi_\Lambda^0(\eta \leqslant n^+)$$

so that

$$\pi_\Lambda^0(A) = \frac{\Pi_\Lambda^0(A)}{1 - \Pi_\Lambda^0\left(\cup_{v\in\Lambda}\{\eta_v > n^+\}\right)} \leqslant \frac{\Pi_\Lambda^0(A)}{1 - \sum_{v\in\Lambda}\Pi_\Lambda^0\left(\{\eta_v > n^+\}\right)}\,.$$

Thanks to monotonicity and Proposition 3.9, for any $v \in \Lambda$ we have

$$\Pi_\Lambda^0(\eta_v > n^+) \leqslant \Pi_\Lambda^{n^+/2}(\eta_v > n^+) \leqslant \frac{\hat\pi_\Lambda^0(\eta_v > n^+/2)}{\hat\pi_\Lambda^0(\eta \geqslant -n^+/2)} \leqslant \frac{ce^{-2\beta n^+}}{1 - c|\Lambda|e^{-2\beta n^+}}\,,$$

(where we took $n^+/2$ to be an integer to simplify the exposition), as required. □

Having bounded the probability of exceeding a certain height in the no floor setting, we can now quantify the entropic repulsion effect and derive an estimate on $\pi_\Lambda^0(\mathscr{C}_{\gamma,h})$.

Proof of Proposition 3.6. Thanks to Lemma 3.2, it suffices to prove the analogous estimates for the measure Π with no ceiling.

By Remark 3.8, the conditional distribution of $\eta\restriction_{\Lambda_\gamma}$ given $\mathscr{C}_{\gamma,h}$ is equal to $\Pi^h_{\Lambda_\gamma}(\cdot \mid \eta\restriction_{\Delta^+_\gamma} \geqslant h)$ which stochastically dominates $\Pi^h_{\Lambda_\gamma}$. Hence,

$$\Pi^0_\Lambda\left(\eta\restriction_{\Lambda_\gamma} > 0 \mid \mathscr{C}_{\gamma,h}\right) \geqslant \Pi^h_{\Lambda_\gamma}(\eta\restriction_{\Lambda_\gamma} > 0) \geqslant \prod_{v\in\Lambda_\gamma} \Pi^h_{\Lambda_\gamma}(\eta_v > 0)$$
$$\geqslant \prod_{v\in\Lambda_\gamma} \hat{\pi}^h_{\Lambda_\gamma}(\eta_v > 0) \geqslant \left(\tfrac{1}{2} \vee (1 - ce^{-4\beta h})\right)^{|\Lambda_\gamma|}$$

where the second inequality follows by the FKG inequality, the third follows by monotonicity of removing the floor and the final inequality by symmetry and Proposition 3.9. Therefore,

$$\Pi^0_\Lambda(\eta\restriction_{\Lambda_\gamma} > 0 \mid \mathscr{C}_{\gamma,h}) \geqslant \exp\left(-2c|\Lambda_\gamma|e^{-4\beta h}\right), \tag{3.10}$$

since $\frac{1}{2} \vee (1-x) \geqslant \exp(-2x)$ for $x \geqslant 0$. With T_γ defined as in (3.6), on the event that γ is an h-contour and $\eta(\Lambda_\gamma) > 0$ we have $T\eta \geqslant 0$ and $\Pi^0_\Lambda(T\eta) = e^{\beta|\gamma|}\Pi^0_\Lambda(\eta)$. It follows from this bijection that

$$1 \geqslant \sum_{\substack{\eta\,:\,\eta\restriction_{\Lambda_\gamma}>0,\\ \mathscr{C}_{\gamma,h}}} \Pi^0_\Lambda(T\eta) = e^{\beta|\gamma|}\Pi^0_\Lambda\left(\eta\restriction_{\Lambda_\gamma} > 0\ ,\ \mathscr{C}_{\gamma,h}\right)$$
$$\geqslant \exp\left(\beta|\gamma| - 2c|\Lambda_\gamma|e^{-4\beta h}\right)\Pi^0_\Lambda(\mathscr{C}_{\gamma,h}), \tag{3.11}$$

with the second inequality by (3.10). Rearranging this establishes (3.3). To obtain (3.4) note first that the proof applies unchanged if $h_s = h$ for all s, i.e., when the family of disjoint contours is of the form $\{(\gamma_s, h)\}_{s\in\mathscr{S}}$, in this case yielding

$$\Pi^0_\Lambda\Bigg(\bigcap_{s\in\mathscr{S}} \mathscr{C}_{\gamma_s,h}\Bigg) \leqslant \exp\Bigg(\sum_{s\in\mathscr{S}}\left(-\beta|\gamma_s| + 2c|\Lambda_{\gamma_s}|e^{-4\beta h}\right)\Bigg).$$

Now take a general family $\{(\gamma_s, h_s)\}_{s\in\mathscr{S}}$ satisfying the hypothesis of the lemma. We proceed by induction over the levels of the contours from top to bottom. If $h_+ = \max_s h_s$ then conditioning on $\bigcap_{s\in\mathscr{S}:h_s<h_+}\mathscr{C}_{\gamma_s,h_s}$ does not affect the conditional distribution of $\eta(\bigcup_{s\in\mathscr{S}:h_s=h_+}\Lambda_\gamma)$ given $\bigcap_{s\in\mathscr{S}:h_s=h_+}\mathscr{C}_{\gamma_s,h_+}$ (as explained in Remark 3.8). Moreover, given that $\bigcap_{s\in\mathscr{S}}\mathscr{C}_{\gamma_s,h_s}$ holds then $T_{h_+}\eta \in \bigcap_{s\in\mathscr{S}:h_s<h_+}\mathscr{C}_{\gamma_s,h}$, where T_{h_+} denotes the composition of the T_{γ_s}'s for all s such that $h_s = h_+$, that is, reducing the height of every site in $\bigcup_{s\in\mathscr{S}:h_s=h_+}\Lambda_{\gamma_s}$ by 1. This implies that

$$\Pi^0_\Lambda\Bigg(\bigcap_{s\in\mathscr{S}:h_s=h_+}\mathscr{C}_{\gamma_s,h_s}\ \Bigg|\ \bigcap_{s\in\mathscr{S}:h_s<h_+}\mathscr{C}_{\gamma_s,h_s}\Bigg) \leqslant \exp\Bigg(\sum_{s\in\mathscr{S}:h_s=h_+}\left(-\beta|\gamma_s| + 2c|\Lambda_{\gamma_s}|e^{-4\beta h_+}\right)\Bigg).$$

The proof is completed by induction. ☐

Proof of Theorem 3.1, Eq. (3.1). It suffices to prove the corresponding bounds for Π. Set $h = H - k$ and $\mathcal{S}_h(\eta) = \{v \in \Lambda : \eta_v = h\}$. For each $A \subseteq \mathcal{S}_h(\eta)$ we can define $U_A : \Omega \to \Omega$ given by

$$(U_A\eta)_v = \begin{cases} \eta_v + 1 & v \notin A \\ 0 & v \in A. \end{cases}$$

To measure the effect of U_A on the Hamiltonian, observe that U_A is equivalent to incrementing each height by 1 followed by decreasing the sites in A by $h+1$. As such, this operation increases the Hamiltonian by at most $|\partial\Lambda| + 4(h+1)|A|$ and so altogether

$$\Pi_\Lambda^0(U_A\eta) \geqslant \exp\left(-4\beta L - 4\beta(h+1)|A|\right)\Pi_\Lambda^0(\eta)\,.$$

Hence,

$$\begin{aligned}\sum_{A\subseteq\mathcal{S}_h(\eta)} \Pi_\Lambda^0(U_A\eta) &\geqslant \exp(-4\beta L)\left(1+e^{-4\beta(h+1)}\right)^{|\mathcal{S}_h(\eta)|}\Pi_\Lambda^0(\eta), \\ &\geqslant \exp\left(-4\beta L + \tfrac12 e^{-4\beta(h+1)}|\mathcal{S}_h(\eta)|\right)\Pi_\Lambda^0(\eta)\,,\end{aligned}$$

since $e^{-4\beta(h+1)} \leqslant 1$ and $(1+x) \geqslant e^{x/2}$ for $0 \leqslant x \leqslant 1$. By construction we have $U_A\eta \neq U_{A'}\eta$ for any $A \neq A'$ with $A, A' \subseteq \mathcal{S}_h(\eta)$. In addition, if $A \subseteq \mathcal{S}_h(\eta)$ and $A' \subseteq \mathcal{S}_h(\eta')$ for some $\eta \neq \eta'$ then $U_A\eta \neq U_{A'}\eta'$ (thanks to the fact that one can recover A from $U_A\eta$ — the sites at level 0 — then proceed to recover η). We can therefore conclude that

$$\begin{aligned}1 &\geqslant \sum_{\eta\,:\,|\mathcal{S}_h(\eta)|\,\geqslant\, e^{-2\beta k}L^2}\ \sum_{A\subseteq\mathcal{S}_h(\eta)} \Pi_\Lambda^0(U_A\eta) \\ &\geqslant \exp\left(-4\beta L + \tfrac12 e^{-4\beta(h+1)}e^{-2\beta k}L^2\right)\Pi_\Lambda^0(|\mathcal{S}_h(\eta)| \geqslant e^{-2\beta k}L^2),\end{aligned}$$

and so, for $k \geqslant 1$

$$\begin{aligned}\Pi_\Lambda^0(|\mathcal{S}_h(\eta)| \geqslant e^{-2\beta k}L^2) &\leqslant \exp\left(4\beta L - \tfrac12 e^{2\beta k - 8\beta}L\right) \\ &\leqslant \frac12 \exp\left(-e^{\beta k}L\right)\,,\end{aligned}$$

where the last inequality holds for any $k \geqslant 12$. A union bound over all $k \geqslant 12$ now holds at the cost of increasing the pre-factor of $1/2$ to 1, as desired. ☐

Proof of Theorem 3.1, Eq. (3.2). As above, we prove the corresponding bounds for Π and the result for π will follow from Lemma 3.2. Let $\mu_2 < 2.68$ be the connective constant in $\mathbb{Z}^2$ and set

$$h = H + \left\lceil \frac{1}{4\beta}\log\left(\frac{C_0}{1-\log\mu_2}\right)\right\rceil$$

where $C_0 > 0$ is the absolute constant from Proposition 3.6. By the isoperimetric inequality in $\mathbb{Z}^2$ we have $|\Lambda_\gamma| \leqslant (L/4)|\gamma|$ for any contour γ in an $L\times L$ box Λ. Plugging these in (3.3) gives

$$\Pi_\Lambda^0\left(\mathscr{C}_{\gamma,h}\right) \leqslant \exp\left(-\beta|\gamma| + C_0(L/4)|\gamma|e^{-4\beta h}\right) \leqslant \exp(-\theta|\gamma|) \tag{3.12}$$

where

$$\theta = \beta - \tfrac14(1-\log\mu_2) \geqslant 1 - \tfrac14(1-\log\mu_2) > \log\mu_2$$

by our hypothesis that $\beta \geqslant 1$.

Now define the random set $\mathscr{A}$ by

$$\mathscr{A} = \mathscr{A}(\eta) = \left\{\gamma : \gamma \text{ is an } h\text{-contour of } \eta \text{ of length } |\gamma| \leqslant \log^2 L\right\}$$

and let $\mathscr{A}_0$ be the result of omitting nested contours from $\mathscr{A}$:

$$\mathscr{A}_0 = \mathscr{A}_0(\eta) = \mathscr{A} \setminus \{\psi \in \mathscr{A} : \Lambda_\psi \subsetneq \Lambda_\gamma \text{ for some } \gamma \in \mathscr{A}\}\,.$$

For any collection of contours A let also

$$E_A = \left\{ \Big| \bigcup_{\gamma\in A} \{v \in \Lambda_\gamma : \eta_v \geqslant h+k\} \Big| > \frac{1}{2} e^{-2\beta k} L^2 \right\}$$

and observe that $E_{\mathscr{A}} = E_{\mathscr{A}_0}$ since $\bigcup\{\Lambda_\gamma : \gamma \in \mathscr{A}\} = \bigcup\{\Lambda_\gamma : \gamma \in \mathscr{A}_0\}$. We thus have

$$\begin{aligned}\Pi^0_\Lambda\left(E_{\mathscr{A}}\right) &= \sum_{A_0} \Pi^0_\Lambda\left(E_{A_0} \mid \mathscr{A}_0 = A_0\right) \Pi^0_\Lambda(\mathscr{A}_0 = A_0) \\ &= \sum_{A_0} \Pi^0_\Lambda\Big(\textstyle\sum_{\gamma\in A_0} \mathscr{X}_\gamma > \frac{1}{2} e^{-2\beta k} L^2 \;\Big|\; \mathscr{A}_0 = A_0 \Big) \Pi^0_\Lambda(\mathscr{A}_0 = A_0) \end{aligned} \tag{3.13}$$

where

$$\mathscr{X}_\gamma = \sum_{v\in\Lambda_\gamma} \mathbf{1}_{\eta_v \,\geqslant\, h+k}\,.$$

Conditioned on $\mathscr{A}_0 = A_0$, monotonicity enables us to increase the values along Δ^-_γ for every $\gamma \in A_0$ to $h-1$ while possibly only increasing the probability of the event E_{A_0}, and by doing so the variables $\{\mathscr{X}_\gamma : \gamma \in \mathscr{A}_0\}$ become mutually independent.

Fix $\gamma \in A_0$. If $\eta_v \geqslant h+k$ for some $v \in \Lambda_\gamma$ this gives rise to a sequence of nested j-contours for $j = h+1, \dots, h+k$ surrounding v, and by Proposition 3.6 the probability for a given fixed such sequence $\psi_1, \dots, \psi_k$ is at most

$$\exp\Big(-\beta \sum_j \left(|\psi_j| + C_0 |\Lambda_{\psi_j}| e^{-4\beta(h+j)} \right) \Big)\,.$$

However, the fact that $\sum_j |\Lambda_{\psi_j}| e^{-4\beta(h+j)} = O(L^{-1}\log^4 L)$ shows that the area term in this estimate is negligible, hence the same argument used for proving the upper bound of Proposition 3.9 (in the no floor setting) yields that, for some absolute $c > 0$ and every $v \in \Lambda_\gamma$,

$$\Pi^0_\Lambda\left(\eta_v \geqslant h+k \mid \mathscr{C}_{\gamma,h}\right) \leqslant c\exp(-4\beta k)\,.$$

In particular,

$$\mathbb{E}_{\Pi^0_\Lambda(\,\cdot\,|\mathscr{C}_{\gamma,h})}\big[\mathscr{X}_\gamma\big] \leqslant |\Lambda_\gamma| c \exp(-4\beta k)\,,$$

and so

$$\mathbb{E}_{\Pi^0_\Lambda(\,\cdot\,|\cap_{\gamma\in A_0}\mathscr{C}_{\gamma,h})}\Big[\sum_{\gamma\in A_0} \mathscr{X}_\gamma\Big] \leqslant c\exp(-4\beta k) \sum_{\gamma\in A_0} |\Lambda_\gamma| \leqslant c\exp(-4\beta k) L^2\,.$$

The variable $\mathscr{Y} = \sum_{\gamma\in A_0} \mathscr{X}_\gamma$ is therefore a sum of $|A_0| \leqslant L^2$ independent variables, each of which respects the bound $|\mathscr{X}_\gamma| \leqslant |\Lambda_\gamma| \leqslant \log^4 L$ with probability 1. Since $\beta \geqslant 1$, for any $k \geqslant \frac{1}{2}\log(4c)$ we have $\mathbb{E}_{\Pi^0_\Lambda(\,\cdot\,|\cap_{\gamma\in A_0}\mathscr{C}_{\gamma,h})}\big[\mathscr{Y}\big] \leqslant \frac{1}{4} e^{-2\beta k} L^2$, and applying Hoeffding-Azuma now gives

$$\Pi^0_\Lambda\Big(\mathscr{Y} \geqslant \tfrac{1}{2} e^{-2\beta k} L^2 \;\Big|\; \mathscr{A}_0 = A_0\Big) \leqslant \exp\Big(-\tfrac{1}{32} e^{-4\beta k} L^2 \log^{-8} L\Big)\,.$$

Together with (3.13) we finally get

$$\Pi^0_\Lambda\left(E_{\mathscr{A}}\right) \leqslant \exp\Big(-e^{-4\beta k} L^{2-o(1)}\Big)\,. \tag{3.14}$$

Having accounted for this probability in the inequality (3.2), we are left with the problem of handling the contribution of long contours, namely those whose length exceeds $\log^2 L$.

Set

$$\mathscr{B} = \mathscr{B}(\eta) = \big\{\gamma : \gamma \text{ is an } h\text{-contour of } \eta \text{ of length } |\gamma| > \log^2 L\big\}\,.$$

We have shown in (3.12) that, for some $\theta \geqslant \theta_0$ with a fixed $\theta_0 > \log \mu_2$ and any given contour γ,

$$\Pi^0_\Lambda\left(\mathscr{C}_{\gamma,h}\right) \leqslant \exp(-\theta|\gamma|)\,.$$

By the same argument (appealing to Proposition 3.6, this time to the more general bound (3.5)), if, for some $m = m(L)$, one considers contours $\gamma_1, \dots, \gamma_m$ with disjoint interiors $\{\Lambda_{\gamma_i}\}_{i=1}^m$ and individual lengths all exceeding $\log^2 L$, then

$$\Pi^0_\Lambda\bigg(\bigcap_{i=1}^m \mathscr{C}_{\gamma_i,h}\bigg) \leqslant \exp\bigg(-\theta\sum_{i=1}^m |\gamma_i|\bigg)\,.$$

By enumerating over the length of each contour γ_i, then selecting its origin and a self-avoiding path for it (the number of options for the latter being counted by $R_{|\gamma_i|}$), we see that

$$\Pi^0_\Lambda\bigg(\bigcup_m \bigcup_{\{\gamma_i\}_{i=1}^m} \bigcap_{i=1}^m \mathscr{C}_{\gamma_i,h}\bigg) \leqslant \sum_m \prod_{i=1}^m \sum_{\log^2 L < |\gamma_i| \leqslant L^2} L^2 R_{|\gamma_i|} e^{-\theta|\gamma_i|}\,.$$

The relation between θ and $\log \mu_2$ suffices to eliminate $R_{|\gamma_i|}$ while still retaining a factor of $\exp(-c\sum_i |\gamma_i|)$ for some absolute $c > 0$. The fact that $\sum_i |\gamma_i|$ is super-logarithmic now eliminates the L^2 pre-factor, as well as the additional enumeration over m itself (another polynomial factor). Altogether,

$$\Pi^0_\Lambda\bigg(\sum_{\gamma\in\mathscr{B}} |\gamma| \geqslant \tfrac{1}{2} e^{-2\beta k} L\bigg) \leqslant \exp\Big(-c e^{-2\beta k} L\Big)$$

for some absolute $c > 0$, and in particular (via the isoperimetric inequality $|\Lambda_\gamma| \leqslant (L/4)|\gamma|$)

$$\Pi^0_\Lambda\bigg(\Big|\bigcup_{\gamma\in\mathscr{B}} \{v \in \Lambda_\gamma : \eta_v \geqslant h\}\Big| > \tfrac{1}{8} e^{-2\beta k} L^2\bigg) \leqslant \exp\Big(-c e^{-2\beta k} L\Big)\,. \tag{3.15}$$

Together with the aforementioned bound on $\Pi^0_\Lambda(E_{\mathscr{A}})$, this completes the proof. ☐

4. Lower bounds on equilibration times

4.1. Proof of Theorem 1: lower bound on the mixing time.

Set

$$h = H - K$$

where K is the constant from Theorem 3.1, and define

$$\mathcal{B} = \left\{\eta : \#\{x \in \Lambda_L :\ \eta_x \geqslant h+1\} \geqslant \frac{1}{2} L^2\right\}\,.$$

Note that, since $\exp(-2\beta K) \leqslant \frac{1}{2}$, Eq. (3.1) of Theorem 3.1 implies that

$$\pi^0_\Lambda\left(\mathcal{B}\right) = 1 - o(1)\,. \tag{4.1}$$

Hence, if $\tau_{\mathcal{B}}$ denotes the hitting time of the set $\mathcal{B}$, it will suffice to show that for a sufficiently small constant $c > 0$

$$\min_\eta \mathbb{P}^\eta(\tau_{\mathcal{B}} < e^{cL}) = o(1). \tag{4.2}$$

For this purpose we observe that $\mathcal{B}$ is an increasing event so that,

$$\min_\eta \mathbb{P}^\eta(\tau_{\mathcal{B}} < e^{cL}) = \mathbb{P}^{\sqcup}(\tau_{\mathcal{B}} < e^{cL}) \leqslant \mathbb{P}^{\nu}(\tau_{\mathcal{B}} < e^{cL})$$

for any initial law ν.

We now choose ν as follows. Take $\delta \in (0, \frac{1}{4})$ to be a sufficiently small constant so that in terms of the constant C_0 from (3.3)

$$\delta < \left[(\beta - \log \mu_2) \frac{4}{C_0} \exp\left(-4\beta(H - h + 1)\right)\right]^2 ,$$

where μ_2 is the connective constant in $\mathbb{Z}^2$. Rearranging the above condition gives

$$\lambda := \sqrt{\delta}(C_0/4) \exp\left(4\beta(H - h + 1)\right) < \beta - \log \mu_2 \,. \tag{4.3}$$

Then we take as starting law ν the conditional measure $\pi^0_\Lambda(\cdot \mid A)$ where A is the event that there exists no h-contour γ with area exceeding δL^2 i.e.,

$$A = \bigcap_{\gamma : |\Lambda_\gamma| > \delta L^2} (\mathscr{C}_{\gamma,h})^c.$$

In the sequel ∂A will denote the internal boundary of A defined by

$$\partial A := \{\eta \in A : \; p(\eta, \eta') > 0 \text{ for some } \eta' \notin A\}$$

where $p(\cdot, \cdot)$ is the transition probability of the dynamics. Let $\tau_{\partial A}$ be the hitting time of ∂A.

Notice that, up to time $\tau_{\partial A}$, the Glauber dynamics started in $A \setminus \partial A$ coincides with the reflected Glauber dynamics in A whose reversible measure is precisely $\nu \equiv \pi^0_\Lambda(\cdot \mid A)$. Therefore a simple union bound over times $t \in [0, e^{cL}]$ gives that

$$\begin{aligned} \mathbb{P}^\nu(\tau_{\mathcal{B}} < e^{cL}) &\leqslant \mathbb{P}^\nu(\tau_{\partial A} < e^{cL}) + \mathbb{P}^\nu(\tau_{\mathcal{B}} < e^{cL} \leqslant \tau_{\partial A}) \\ &\leqslant e^{cL} \left(\nu(\partial A) + \nu(\mathcal{B})\right). \end{aligned} \tag{4.4}$$

Define now

$$\tilde{A} = \bigcap_{\gamma : |\Lambda_\gamma| > \frac{1}{5}\delta L^2} (\mathscr{C}_{\gamma,h})^c.$$

Notice that $\partial A \subset A \setminus \tilde{A}$ since at most four distinct h-contours can be combined by the modification of a single site. Therefore

$$\nu(\partial A) = \frac{\pi^0_\Lambda(\partial A)}{\pi^0_\Lambda(A)} \leqslant \frac{\pi^0_\Lambda(A \setminus \tilde{A})}{\pi^0_\Lambda(A)} \,.$$

We next claim that

$$\frac{\pi^0_\Lambda(A \setminus \tilde{A})}{\pi^0_\Lambda(A)} \leqslant e^{-c_1 L} \tag{4.5}$$

for some constant $c_1 = c_1(\beta)$. Indeed, suppose that γ is a contour such that $|\Lambda_\gamma|/L^2 \in (\frac{1}{5}\delta, \delta)$. As in the proof of Proposition 3.6 (see formula (3.11)) and with T_γ defined as in (3.6),

$$\pi^0_\Lambda(A) \geqslant \sum_{\substack{\eta \in A,\, \eta\restriction_{\Lambda_\gamma} > 0 \\ \mathscr{C}_{\gamma,h}}} \pi^0_\Lambda(T_\gamma \eta) = e^{\beta|\gamma|} \, \pi^0_\Lambda(\eta\restriction_{\Lambda_\gamma} > 0 \mid A, \mathscr{C}_{\gamma,h}) \pi^0_\Lambda(A \cap \mathscr{C}_{\gamma,h})$$

where we used the fact that $T_\gamma \eta \in A$ if $\eta \in A \cap \mathscr{C}_{\gamma,h}$. Next we observe that, thanks to (3.10) (which holds with identical proof also for π^0_Λ),

$$\pi^0_\Lambda(\eta\restriction_{\Lambda_\gamma} > 0 \mid A, \mathscr{C}_{\gamma,h}) = \pi^0_\Lambda(\eta\restriction_{\Lambda_\gamma} > 0 \mid \mathscr{C}_{\gamma,h}) \geqslant \exp(-2c|\Lambda_\gamma|e^{-4\beta h})$$

to yield

$$\pi^0_\Lambda\left(\mathscr{C}_{\gamma,h} \mid A\right) \leqslant \exp\left(-\beta|\gamma| + C_0|\Lambda_\gamma| \exp(-4\beta h)\right) . \tag{4.6}$$

The isoperimetric inequality in $\mathbb{Z}^2$ gives that $|\Lambda_\gamma| \leqslant |\gamma|^2/16$ for any γ, so that, by the above choice of parameters, any contour γ with area less than δL^2 satisfies

$$\begin{aligned} C_0|\Lambda_\gamma|e^{-4\beta h} &\leqslant C_0\left(\sqrt{\delta L^2}\sqrt{|\gamma|^2/16}\right)\left(\frac{e^{4\beta H}}{e^{-4\beta}L}\right)e^{-4\beta h} \\ &\leqslant \sqrt{\delta}(C_0/4)|\gamma|e^{4\beta(H-h+1)} = \lambda|\gamma| \end{aligned}$$

where λ is given by (4.3). Hence the r.h.s. of (4.6) is smaller than $e^{-(\beta-\lambda)|\gamma|}$. A union bound over γ's with $|\Lambda_\gamma| > (\delta/5)L^2$ then proves (4.5).

In conclusion the first term in the r.h.s. of (4.4) is $o(1)$ if $c < c_1$. We now examine the second term $\nu(\mathcal{B})$ and we proceed as in the proof of Theorem 3.1. First, we claim that for any short h-contour γ and $v \in \Lambda_\gamma$, where "short" means of length smaller than $\log^2(L)$, we have

$$\Pi^0_\Lambda\left(\eta_v \geqslant h+1 \mid \mathscr{C}_{\gamma,h}\right) \leqslant \tfrac14\,, \tag{4.7}$$

Indeed, if $\mathscr{A} = \{\gamma' : v \in \Lambda_{\gamma'} \subseteq \Lambda_\gamma\}$ then an application of (3.5) from Lemma 3.7 shows that

$$\hat\pi^0_\Lambda\left(\eta_v \geqslant h+1 \mid \mathscr{C}_{\gamma,h}\right) \leqslant \sum_{\gamma'\in\mathscr{A}} \hat\pi^0_\Lambda\left(\mathscr{C}_{\gamma',h+1} \mid \mathscr{C}_{\gamma,h}\right) \leqslant \sum_{\gamma'\in\mathscr{A}} e^{-\beta|\gamma'|} \leqslant \tfrac18$$

for β large since, as usual, the number of contours $\gamma' \in \mathscr{A}$ of length k is at most $k\mu_2^k$ (using the fact that each of these crosses the horizontal line to the right of v within distance at most k). To transfer this estimate to the setting of a floor, observe that by Remark 3.8,

$$\Pi^0_\Lambda\left(\eta_v \geqslant h+1 \mid \mathscr{C}_{\gamma,h}\right) = \frac{\hat\pi^0_\Lambda\left(\eta_v \geqslant h+1\,,\, \eta\restriction_{\Lambda_\gamma} \geqslant 0 \mid \mathscr{C}_{\gamma,h}\right)}{\hat\pi^h_{\Lambda_\gamma}\left(\eta\restriction_{\Lambda_\gamma} \geqslant 0 \mid \eta\restriction_{\Delta^+_\gamma} \geqslant h\right)}\,. \tag{4.8}$$

We have just established that the numerator is at most 1/8, whereas by monotonicity the denominator is at least

$$\hat\pi^h_{\Lambda_\gamma}\left(\eta\restriction_{\Lambda_\gamma} \geqslant 0\right) = \hat\pi^0_{\Lambda_\gamma}\left(\eta\restriction_{\Lambda_\gamma} \geqslant -h\right) \geqslant 1 - ce^{-4\beta(h+1)}|\Lambda_\gamma|$$

thanks to Proposition 3.9 (with the same constant $c > 0$ appearing there) and a union bound over the sites of Λ_γ. The fact that $|\Lambda_\gamma| \leqslant |\gamma|^2 = O(\log^4 L)$ shows this last term is $1 - L^{-1+o(1)}$, hence the effect of the denominator in (4.8) can easily be countered by a factor of 2, thus establishing (4.7).

With inequality (4.7) available to us, the very same concentration argument leading to (3.14) applies again here to imply that

$$\Pi^0_\Lambda\Big({\sum_\gamma}' \#\{x \in \Lambda_\gamma :\ \eta_x \geqslant h+1\} \geqslant \frac12 L^2\Big) \leqslant e^{-c_2 L^{2-o(1)}} \tag{4.9}$$

for some constant $c_2 > 0$, where the summation $\sum'_\gamma$ is over every *short* h-contour γ. Similarly, following the same steps leading to (3.15), we get that

$$\Pi^0_\Lambda\Big({\sum_\gamma}'' \#\{x \in \Lambda_\gamma :\ \eta_x \geqslant h+1\} \geqslant \frac12 L^2\Big) \leqslant e^{-c_3 L} \tag{4.10}$$

for a suitable $c_3 > 0$, where $\sum''_\gamma$ sums over every *long* h-contour γ, i.e. such that $|\gamma| \geqslant (\log L)^2$ (in this case, the analog of (3.12) for h-contours of area smaller than δL^2 holds if δ chosen small). Finally, Lemma 3.2 translates the statements on Π^0_Λ into the analogous bounds for π^0_Λ. In conclusion the second term in the r.h.s. of (4.4) is $o(1)$ if $c < \min(c_2, c_3)$, as required. ☐

4.2. **Proof of Theorem 2: lower bound on τ_a.** Here we prove that $\mathbb{P}^{\sqcup}(\tau_a \geqslant e^{cL^a}) \to 1$ as $L \to \infty$ where, we recall,

$$\Omega_a = \{\eta \text{ such that } \#\{x \in \Lambda_L : \eta_x \geqslant aH(L)\} > \tfrac{9}{10}|\Lambda_L|\}$$

and τ_a is the hitting time of Ω_a. We proceed as in the proof of (4.2) but now the height h is chosen equal to $aH(L) - 1$, so that $e^{-4\beta h} \leqslant \exp(8\beta)L^{-a}$, and the set A is defined by

$$A = \bigcap_{\gamma\,:\,|\Lambda_\gamma| > \delta L^{2a}} (\mathscr{C}_{\gamma,h})^c \,.$$

Here δ is a small constant such that, for $|\Lambda_\gamma| \leqslant \delta L^{2a}$:

$$C_0 |\Lambda_\gamma| e^{-4\beta h} \leqslant C_0 \left(\sqrt{\delta L^{2a}}\sqrt{|\gamma|^2/16}\right) e^{-4\beta h} \leqslant \lambda|\gamma|$$

where λ is analogous to (4.3). As in (4.4) we get

$$\begin{aligned} \mathbb{P}^{\nu}(\tau_a < e^{cL^a}) &\leqslant \mathbb{P}^{\nu}(\tau_{\partial A} < e^{cL^a}) + \mathbb{P}^{\nu}(\tau_a < e^{cL^a} \leqslant \tau_{\partial A}) \\ &\leqslant e^{cL^a} \left(\nu(\partial A) + \nu(\Omega_a)\right) \,. \end{aligned} \tag{4.11}$$

Exactly the same arguments behind (4.5), (4.9) and (4.10) now show that the r.h.s. of (4.11) is $o(1)$. ☐

5. A bound using paths and flows

5.1. **Proof of Proposition 2.3.** Let $\Lambda := \{1, \dots, L\} \times \{1, \dots, m\}$ and $\Omega := \Omega_{\Lambda, n^+}$. We introduce the canonical paths $\gamma(\eta, \eta')$ from η to η' for every $\eta, \eta' \in \Omega$. Define the diagonal lines in $\Lambda_L = \{1, \dots, L\}^2$

$$R_i = \{x \in \Lambda_L : \; x_2 = x_1 + L - i\}, \qquad i = 1, \dots, 2L - 1 \tag{5.1}$$

and let $\mathcal{R}$ denote the collection of the R_i. Number the sites in Λ following the lines $R_1, \dots, R_{2L-1}$, so that each line is read from south-west to north-east; at each site x move straight from η_x to η'_x by taking $|\eta_x - \eta'_x|$ unit steps. Note that since all heights satisfy $0 \leqslant \eta_x \leqslant n^+$ one has $|\gamma| \leqslant |\Lambda| n^+$. If $e = (\sigma, \sigma^{x_*,\pm})$ is an edge of a path, with $x_* \in R_{i_*}$, define A as the set of $x \in \Lambda$ such that $x < x_*$ and B the set of $x > x_*$ (w.r.t. to the order introduced above). Here $\sigma^{x_*,\pm}$ denotes the configuration which coincides with σ except that the height at x_* is changed by ± 1. Then by direct inspection one finds that for any $\eta, \eta' \in \Omega$ such that $\gamma(\eta, \eta') \ni e$:

$$\pi(\eta)\pi(\eta') \leqslant \pi(\sigma)\pi(\sigma^*) \exp\big(6\beta \sum_{x \in R_{i_*} \cap \Lambda} |\eta_x - \eta'_x|\big) \tag{5.2}$$

where σ satisfies $\sigma_A = \eta'_A$, $\sigma_B = \eta_B$, while σ^* is the configuration obtained by setting $\sigma^*_A = \eta_A$, $\sigma^*_B = \eta'_B$. Here σ_{x_*} and $\sigma^*_{x_*} = \sigma_{x_*} \pm 1$ are assigned according to the choice of e. The crucial observation is that, given e, the map from (η, η') (such that $e \in \gamma(\eta, \eta')$) to (σ, σ^*) is an injective one. In particular, this implies:

$$\frac{1}{\pi(\sigma)} \sum_{\eta, \eta' \in \Omega} |\gamma(\eta, \eta')| \pi(\eta)\pi(\eta') \mathbf{1}_{e \in \gamma} \leqslant |\Lambda| n^+ \exp\left(6\beta n^+ m\right). \tag{5.3}$$

Note also that the inverse of the smallest non-zero one-step transition probability for our chain is $|\Lambda| \exp(4\beta n^+)$. We apply then (2.10) to obtain that the inverse spectral gap of the SOS dynamics is upper bounded by $c|\Lambda|^2 n^+ \exp\left(7\beta n^+ m\right)$ and (2.11) follows. ☐

5.2. **Proof of Theorem 2.4.** For every $\xi, \xi' \in \Omega$, let γ_1 be a path of length T starting at ξ and let γ_2 be a path of length T starting at ξ'. Write η, η' for the corresponding endpoints. Let γ_c be a path from η to η' (to be specified below) which depends only on η, η' and not on γ_1, γ_2. Call γ the concatenation of $\gamma_1, \gamma_c, \bar{\gamma}_2$, where $\bar{\gamma}_2$ is the path γ_2, inverted in time. Note that γ connects ξ to ξ'. If $\eta, \eta' \in G$ then we let γ_c be the path $\widetilde{\gamma}(\eta, \eta')$ which appears in the statement of the theorem (recall that it stays in the set G) and define

$$a(\gamma) = \frac{\pi(\xi)\mathbb{P}^{\xi}(\gamma_1)}{\mathbb{P}^{\xi}(X(T) \in G)} \frac{\pi(\xi')\mathbb{P}^{\xi'}(\gamma_2)}{\mathbb{P}^{\xi'}(X(T) \in G)}$$

Otherwise, set $a(\gamma) = 0$ and we do not need to specify γ_c in this case. Here $\mathbb{P}^{\xi}(\gamma_1)$ is the probability that the process $(X(t))_t$ started at ξ follows exactly γ_1 up to time T, and similar for $\mathbb{P}^{\xi'}(\gamma_2)$. Note that for fixed $\xi, \xi' \in \Omega$, $\sum_{\gamma:\xi\sim\xi'} a(\gamma) = \pi(\xi)\pi(\xi')$ where the sum is over $\eta, \eta', \gamma_1, \gamma_2$ for fixed ξ, ξ'.

Therefore, viewing the path γ as a collection of oriented edges $e = (\sigma, \sigma')$ and letting $\nabla_e f = f(\sigma) - f(\sigma')$, we have

$$\mathrm{Var}(f) = \frac{1}{2}\sum_{\xi,\xi'} \pi(\xi)\pi(\xi')(f(\xi) - f(\xi'))^2 = \frac{1}{2}\sum_{\xi,\xi'}\sum_{\gamma:\xi\sim\xi'} a(\gamma)\Big(\sum_{e\in\gamma} \nabla_e f\Big)^2 \tag{5.4}$$

$$\leqslant \frac{3}{2}\sum_{\xi,\xi'}\sum_{\gamma:\xi\sim\xi'} a(\gamma)(A_\gamma(f) + B_\gamma(f)), \tag{5.5}$$

where

$$A_\gamma(f) = |\gamma_1| \sum_{e\in\gamma_1} (\nabla_e f)^2 + |\gamma_2| \sum_{e\in\gamma_2} (\nabla_e f)^2$$

$$B_\gamma(f) = |\widetilde{\gamma}| \sum_{e\in\widetilde{\gamma}} (\nabla_e f)^2$$

and in the inequality we used Cauchy-Schwarz. Now we use the fact that the Dirichlet form which appears in the definition (2.5) of the spectral gap can be written as

$$\mathcal{E}(f) := \pi^0_\Lambda(f(I - P)f) = \frac{1}{2}\sum_{e=(\sigma,\sigma')} \pi(\sigma)p(\sigma,\sigma')(\nabla_e f)^2\,.$$

Recall that $p_{\min}$ denotes the smallest non-zero one-step transition probability, and observe that

$$\sum_{\xi,\xi'}\sum_{\gamma:\xi\sim\xi'} a(\gamma)A_\gamma(f) = 2\sum_{\xi}\sum_{\gamma_1} |\gamma_1| \frac{\pi(\xi)\mathbb{P}^{\xi}(\gamma_1)}{\mathbb{P}^{\xi}(X(T) \in G)} \sum_{e\in\gamma_1} (\nabla_e f)^2$$

$$\leqslant 4\frac{T}{\alpha\, p_{\min}}\,\mathcal{E}(f) \sup_{e=(\sigma,\sigma')} (\pi(\sigma)^{-1}) \sum_{\xi}\sum_{\gamma_1} \pi(\xi)\mathbb{P}^{\xi}(\gamma_1)\mathbf{1}_{e\in\gamma_1}\,,$$

where we used the fact that $|\gamma_1| = T$.

Let $\mathbb{P}$ denote the law of the stationary process (started at equilibrium π). From a union bound, one has

$$\sum_{\xi}\sum_{\gamma_1} \pi(\xi)\mathbb{P}^{\xi}(\gamma_1)\mathbf{1}_{e\in\gamma_1} = \mathbb{P}(\exists t \in [0,T]:\ X(t) = \sigma,\ X(t+1) = \sigma') \leqslant T\pi(\sigma).$$

It then follows that

$$\sum_{\xi,\xi'}\sum_{\gamma:\xi\sim\xi'} a(\gamma)A_\gamma(f) \leqslant 4\frac{T^2}{\alpha p_{\min}}\,\mathcal{E}(f).$$

As for the second term in (5.5), using stationarity of π one has that the sum of $\pi(\xi)\mathbb{P}^\xi(\gamma_1)$ over all ξ and paths γ_1 of length T which connect ξ to η gives $\pi(\eta)$, so that (with the definition (2.12))

$$\sum_{\xi,\xi'}\sum_{\gamma:\xi\sim\xi'} a(\gamma)B_\gamma(f) \leqslant \frac{2}{\alpha^2}\frac{1}{2}\sum_{e=(\sigma,\sigma')}(\nabla_e f)^2\pi(\sigma)p(\sigma,\sigma')\sum_{\eta,\eta'\in G}\frac{|\widetilde{\gamma}(\eta,\eta')|\pi(\eta)\pi(\eta')}{\pi(\sigma)p(\sigma,\sigma')}\mathbf{1}_{e\in\widetilde{\gamma}(\eta,\eta')}$$
$$\leqslant \frac{2}{\alpha^2}W(G)\,\mathcal{E}(f).$$

Going back to (5.5) and to the definition of spectral gap one immediately gets (2.13). □

6. Upper bounds on equilibration times

6.1. **Proof of Theorem 2: upper bound on τ_a assuming Theorem 1.** Here we prove that $\mathbb{P}^{\sqcup}(\tau_a \leqslant e^{c'L^a}) \to 1$ as $L\to\infty$ assuming $T_{\text{MIX}} \leqslant e^{cL}$. The latter estimate will be proven afterwards. Let us partition the box Λ_L into non-overlapping squares Q_i of side CL^a with $C = \exp(4\beta K)$ where K is the constant appearing in Theorem 3.1. By monotonicity the Glauber dynamics is higher than the auxiliary dynamics in which each square Q_i evolves independently from the others with 0 boundary conditions on ∂Q_i. Using the assumption $T_{\text{MIX}} \leqslant e^{cL}$ and independence, it is standard to check that the mixing time of this auxiliary dynamics is not larger than e^{2cL^a} and therefore, at time $T = e^{3cL^a}$, all the squares Q_i are close to their equilibrium (in total variation) with an exponentially small error. Theorem 3.1 implies that in each of them the density of vertices higher than

$$H(CL^a) - K = aH(L)$$

is larger than $1-\varepsilon(\beta)$ with probability exponentially close to one. In conclusion, apart from an exponentially small error, $\mathbb{P}^{\sqcup}(\tau_a > e^{3cL^a})$ is bounded by the probability that for some i the square Q_i has a density less that $1-\varepsilon(\beta)$ of vertices higher than $H(CL^a)-K$. Thus, a union bound suffices to conclude the proof. □

6.2. **Proof of $T_{\text{mix}} \leqslant e^{cL}$ for $n^+ = \log L$.** To prove the upper bound on T_{MIX} in Theorem 1, the crucial point is to give the proof for $n^+ = \log L$, so we assume this is the case in this section. The general case $\log L \leqslant n^+ \leqslant L$ can be then deduced via very soft arguments; see Section 6.3 below.

For reasons that will be clear later, first of all we modify the SOS model by considering the Boltzmann factor $\exp[-\beta\mathcal{H}^\xi_{\Lambda_L} + f]$ instead of $\exp[-\beta\mathcal{H}^\xi_{\Lambda_L}]$, where f is the external field term

$$f = \frac{1}{L}\sum_{y\in\Lambda_L} f_y, \qquad \text{with} \qquad f_y = \sum_{j=1}^{n^+-H} f_{y,j} := \sum_{j=1}^{n^+-H} c_j \mathbf{1}_{\eta_y \leqslant H+j}, \tag{6.1}$$

with $H = H(L)$ defined in (1.3) and $c_j = \exp(-\beta j)$. One changes the partition function accordingly. We call $\pi^{\xi,f}_{\Lambda_L}$ the corresponding equilibrium measure with ceiling at $n^+ = \log L$ and floor at 0. Moreover, we will consider the Glauber (heat bath) dynamics associated to $\pi^{\xi,f}_{\Lambda_L}$.

Remark 6.1. *Note that, if the b.c. are zero then the extra term f in (6.1) will not drastically change the global equilibrium properties, since it tends to depress the heights that exceed the level H (and having $\eta_x \geqslant H+1$ is already an unlikely event, for β large). More precisely, f equals the constant $(|\Lambda_L|/L)\sum_j c_j$ plus a (negative) random term which one could prove, by refining the estimates of Section 3, to be of order $L\times\exp(-c\,\beta)$ for a typical configuration (and therefore not extensive in the area of Λ_L).*

The reason for modifying the equilibrium measure in such a peculiar way is explained after Theorem 6.12.

Lemma 6.2. *The ratio Δ of the mixing time of the original system over the mixing time of the system modified as in (6.1) satisfies for L large*

$$e^{-L} \leqslant \Delta \leqslant e^{L}. \tag{6.2}$$

Proof. Going back to the definition (2.5) of the spectral gap, it is easy to see that the ratio $\tilde{\Delta}$ of relaxation times satisfies

$$e^{-4|f|_\infty} \leqslant \tilde{\Delta} \leqslant e^{4|f|_\infty}$$

with f as in (6.1); see e.g. [39, Lemma 13.22] for such standard comparison bounds. Note that $|f|_\infty = O(Le^{-\beta})$ if β is large enough. Then (6.2) follows from the comparison (2.7). □

Therefore, it is enough to prove Theorem 1 for this modified model. We denote its mixing time as $T_{\text{MIX}}(L)$. It is important to realize that the Glauber dynamics for this modified SOS model is still monotone (in the sense of Section 2.2) and that the FKG inequalities are still valid. This is because f is the sum of functions of a single height η_x. Therefore we can apply all the monotonicity arguments we need (including the Peres-Winkler censoring inequality, Theorem 2.2).

Definition 6.3. *For $k \in \mathbb{N}$ and $a, A > 0$, we define the inductive statement $\mathcal{F}_k := \mathcal{F}_{k,a,A}$: for every L the mixing time satisfies*

$$T_{\text{mix}}(L) \leqslant L^a e^{AL \log^{(k)}(L)}$$

where $\log^{(k)}(x) := \max(1, \log(\log ...(x)))$ and $\log(\log ...(x))$ is the logarithm iterated k times.

Theorem 6.4. *Fix $\beta \geqslant \beta_0$ for some large enough constant β_0, and $n^+ = \log L$. Then $\mathcal{F}_k \Rightarrow \mathcal{F}_{k+1}$ provided that $a = 4$ and $A = C\beta$ for some sufficiently large C.*

Proof of Theorem 1 given Theorem 6.4. For $k = 1$ the statement $\mathcal{F}_1$ follows at once from the "canonical paths argument", Proposition 2.3 (with $a = 3$ and $A = b\beta$, b some explicit constant). Notice that Proposition 2.3 applies with no change to the modified model with the external field. Then, apply the theorem until $\log^{(k)}(L) = 1$. At that point we get the desired exponential mixing time upper bound. □

For the proof of Theorem 6.4, we need some notation. Recall the definition (5.1) of the diagonal lines R_i. Define $G_\ell^+ \subset \Omega_L$ as the set of configurations η such that, for every $R \in \mathcal{R}$,

$$\sum_{x\in R} [\eta_x - H]^+ \leqslant L\ell$$

(with $[x]^+ = \max(x, 0)$), $G_\ell^- \subset \Omega_L$ as the set of configurations η such that, for every $R \in \mathcal{R}$,

$$\sum_{x\in R} [H - \eta_x]^+ \leqslant L\ell$$

and finally $G_\ell \subset \Omega_L$ as the set of configurations η such that, for every $R \in \mathcal{R}$,

$$\sum_{x\in R} |H - \eta_x| \leqslant L\ell. \tag{6.3}$$

Let also

$$\ell(k, L) := B \log^{(k)}(L) + \frac{1}{4\beta} \log A \tag{6.4}$$

with B a constant to be chosen sufficiently large (independently of β) later, see discussion after (6.7) and (6.20).

Lemma 6.5. *Assume $\mathcal{F}_k$ with $a=4$ and $A=40B\beta$ and take $T_1 > e^{2L}$. Then*

$$\mathbb{P}(\eta^{\sqcup}(T_1) \in G^-_{\ell(k+1,L)}) \geqslant \tfrac{3}{4}.$$

Lemma 6.6. *Assume $\mathcal{F}_k$ with $a=4$ and $A=40B\beta$ and take $T_2 > e^{B\beta L}$ with the same B as in (6.4). Then*

$$\mathbb{P}(\eta^{\sqcap}(T_2) \in G^+_{\ell(\infty,L)}) \geqslant \tfrac{3}{4}.$$

Note that $\ell(\infty, L) = B + 1/(4\beta)\log A$ is just a large constant. We will actually see that, in both lemmas, the constant $\frac{3}{4}$ can be replaced by $1-o(1)$ where $o(1)$ vanishes for $L\to\infty$. We refer to Section 6.4 and Section 6.5 below for the proof of Lemma 6.5 and Lemma 6.6 respectively.

Proof of Theorem 6.4 given Lemmas 6.5 and 6.6. Thanks to monotonicity, to Lemmas 6.5–6.6 and the fact that $G^+_{\ell(\infty,L)} \subset G^+_{\ell(k+1,L)}$, we can set $T^{\rm all} := \max\left(e^{2L}, e^{B\beta L}\right)$ and obtain that

$$\min_{\zeta} \mathbb{P}(\eta^{\zeta}(T^{\rm all}) \in G_{2\ell(k+1,L)}) \geqslant \tfrac{1}{2}. \tag{6.5}$$

This is based on the fact that, if $\eta^1 \leqslant \eta \leqslant \eta^2$ and $\eta^1 \in G^-_{\ell}, \eta^2 \in G^+_{\ell'}$ then $\eta \in G_{\ell+\ell'}$. Just write

$$|\eta_x - H| = [\eta_x - H]^+ + [H-\eta_x]^+ \leqslant [\eta^2_x - H]^+ + [H - \eta^1_x]^+.$$

At this point we need the following consequence of Theorem 2.4.

Proposition 6.7. *Let $\alpha, \ell > 0$ and T be such that $\mathbb{P}(\eta^{\zeta}(T) \in G_\ell) \geqslant \alpha$ for all initial configurations ζ. Then there exists a constant $c = c(\alpha,\beta)$ such that*

$$T_{\rm REL}(L) \leqslant c\left[\exp\left(15\beta\ell L\right) + L^{5\beta}\, T^2\right]. \tag{6.6}$$

Proof. Theorem 2.4 gives

$$T_{\rm REL} \leqslant \frac{6}{\alpha}\left(\frac{T^2}{p_{\min}} + \frac{W(G_\ell)}{\alpha}\right)$$

where in the definition of $W(G_\ell)$ we choose the canonical paths introduced in Section 5.1. We know that the inverse of the minimal transition probability $p_{\min}$ is of order $|\Lambda_L|\exp(4\beta n^+)$. Also, from the proof of Proposition 2.3 and the definition (6.3) of G_ℓ, we see easily that $W(G_\ell) \leqslant \exp(15\beta\ell L)$ and then the claim follows. $\square$

Proposition 6.7 (applied with $T = T^{\rm all}$, ℓ replaced by $2\ell(k+1,L)$ and recalling that $n^+ = \log L$), together with (6.5) and (2.7), implies that

$$T_{\rm mix}(L) \leqslant c'(\beta)L^3\left(L^{5\beta}(T^{\rm all})^2 + e^{30B\,\beta L\log^{(k+1)}(L) + 8L\log A}\right). \tag{6.7}$$

If one chooses $A = 40B\beta$ (and B large but independent of k,β) then the r.h.s. of (6.7) is smaller than $L^4\exp(AL\log^{(k+1)}(L))$ for every L and the claim follows. $\square$

6.3. **Proof of $T_{\mathbf{mix}} \leqslant e^{cL}$ for $\log L \leqslant n^+ \leqslant L$.** Once we have the statement for $n^+ = \log L$, proving it for $\log L \leqslant n^+ \leqslant L$ is quite easy, so we only sketch the main steps. Thanks to (2.8), it is enough to prove that

$$\|\mu_t^{\sqcup} - \pi\| \leqslant L^{-4} \tag{6.8}$$

$$\|\mu_t^{\sqcap} - \pi\| \leqslant L^{-4} \tag{6.9}$$

for some $t = \exp(O(\beta L))$. Here we write π instead of π_Λ^0 for simplicity. We first note that, if $\pi, \tilde\pi$ are the equilibria with ceiling at $n^+ > \log L$ and at $\log L$ respectively, then

$$\|\pi - \tilde\pi\| \leqslant L^{-c_0(\beta)} \tag{6.10}$$

with $c_0(\beta)$ that diverges as $\beta \to \infty$. Indeed, to feel the ceiling there must be some x such that $\eta_x \geqslant \log L$ and this has probability at most $c|\Lambda_L|\exp(-2\beta \log L)$. This can be seen as follows. By monotonicity lift the b.c. from 0 to $(\log L)/2$. In this situation, the probability that the SOS interface reaches either height 0 or $\log L$ is $O(|\Lambda_L| e^{-2\beta \log L})$, as follows from Proposition 3.9 and a union bound, cf. the proof of Lemma 3.2.

As for (6.8), from Theorem 2.2 (applied with $k = 1$, $t_1 = t, V_1 = \Lambda_L, a_1 = 0, b_1 = \log L$) we have $\|\mu_t^{\sqcup} - \pi\| \leqslant \|\tilde\mu_t^{\sqcup} - \pi\|$, with $\tilde\mu_t$ the law of the evolution $\tilde\eta(t)$ with ceiling at $\log L$. Since we proved in Section 6.2 that the mixing time of the dynamics $\tilde\eta(t)$ is $\exp(O(\beta L))$, if $t = \exp(c\beta L)$ with c large one gets from (2.4) that $\|\tilde\mu_t^{\sqcup} - \tilde\pi\| = o(L^{-4})$ and therefore $\|\tilde\mu_t^{\sqcup} - \pi\| = O(L^{-c_0(\beta)}) + \|\tilde\mu_t^{\sqcup} - \tilde\pi\| = o(L^{-4})$ if β is large enough.

As for (6.9), assume for definiteness that n^+ is a multiple of $\log L$ and let

$$h_i = n^+ - \frac{i-1}{2}\log L\,, \qquad i = 1, \dots, M := \frac{2n^+}{\log L} - 1\,.$$

Let us apply Theorem 2.2 with $k = M$, $V_i = \Lambda_L$, $t_i = i\exp(c\beta L)$ with c large enough, $b_i = h_i$ and $a_i = h_i - \log L$. Let us also call U_i the event that $a_i \leqslant \eta_x \leqslant a_i + \frac{1}{2}\log L$ for all $x \in \Lambda_L$. Note that, for the associated modified dynamics $\tilde\eta(t)$, in the time interval $0 < t \leqslant t_1 = \exp(c\beta L)$ the floor is at height $a_1 = n^+ - \log L$ and the ceiling at height $b_1 = n^+$. Therefore, if c is chosen large enough, at time $\exp(c\beta L)$ the system is within variation distance say e^{-L} from the equilibrium with such floor/ceiling and in particular, except with probability smaller than $L^{-c_0(\beta)}$, the configuration is in U_1 (the proof of this is very similar to the proof of (6.10) above). If $\tilde\eta(t_1) \in U_1$, then in the second time-lag $\{t_1 + 1, \dots, t_2\}$ the situation is similar, except that the floor is now at a_2 and the ceiling is at b_2 (note that if instead $\tilde\eta(t_1) \notin U_1$ then some heights are frozen forever to values larger than $a_1 + \frac{1}{2}\log L$ and the dynamics $\tilde\eta(t)$ will not approach equilibrium). The argument is repeated M times with the result that (via a union bound on i), at time $t_M = \exp(O(\beta L))$, the variation distance from equilibrium is smaller than $ML^{-c_0(\beta)} \lll L^{-4}$ and the proof is concluded. □

6.4. **Rising from the floor: Proof of Lemma 6.5.** We will make a union bound on $R_i \in \mathcal{R}$, i.e. on $i = 1, \dots, 2L-1$. We want to upper bound

$$\mathbb{P}\left(\sum_{x \in R_i} [H - \eta_x^{\sqcup}(T_1)]^+ \geqslant L\ell(k+1, L)\right). \tag{6.11}$$

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In principle the argument is very simple. Around every point $x \in R_i$ one would like to consider a square Q_x of side $L_k := L/(A\log^{(k)}(L))$. By monotonicity the quantity $[H - \eta_x^{\sqcup}(T_1)]^+$ appearing in (6.15) gets larger if we fix to 0 the heights on ∂Q_x. From the assumption $\mathcal{F}_k$ we know that the mixing time in Q_x, with zero b.c.

on ∂Q_x, is of order $\exp(AL_k \log^{(k)}(L_k)) \approx \exp(L) \ll T_1 \approx \exp(2L)$. Thus, at time T_1 the dynamics in Q_x is essentially at equilibrium (w.r.t. zero b.c. on ∂Q_x), so that $\eta_x \sim 1/(4\beta)\log L_k \approx H - (1/4\beta)\log^{(k+1)}(L)$ w.h.p. By taking the constant B appearing in (6.4) large enough we can make $\ell(k+1,L) \gg (1/4\beta)\log^{(k+1)}(L)$. As a consequence, the event in (6.11) describes a very unlikely deviation.

In practice, the proof is considerably more involved, in particular because the size of the squares Q_x has to be chosen as a function of x (cf. (6.16)) in order to guarantee that Q_x is fully contained in the original domain Λ_L.

Set $\sigma_x := \frac{1}{4\beta}\log d(x)$, where $d(x)$ is the L^1 distance of x from the boundary of Λ_L. One has

$$[H - \eta_x^{\sqcup}(T_1)]^+ \leqslant [\sigma_x - \eta_x^{\sqcup}(T_1)]^+ + |\sigma_x - H|. \tag{6.12}$$

Now, there exists C_1 such that for every $R_i \in \mathcal{R}$ one has

$$\sum_{x\in R_i} |\sigma_x - H| \leqslant \frac{C_1}{\beta} L. \tag{6.13}$$

By the way, this is the reason why we defined the lines R_i as in (5.1): if R_i were parallel to the coordinate axes and too close to the boundary of Λ_L, then (6.13) would be false. To prove (6.13), suppose without loss of generality that the diagonal line under consideration is R_i with $i \leqslant L$, so that $|R_i| = i$. One has

$$\sum_{x\in R_i} |\sigma_x - H| = iH - \sum_{x\in R_i}\sigma_x \leqslant \frac{1}{4\beta} i\Big(\log L - \frac{1}{i}\sum_{x\in R_i}\log d(x)\Big). \tag{6.14}$$

If i is even

$$\sum_{x\in R_i}\log d(x) = 2\sum_{k=1}^{i/2}\log k = i\log i + O(i)$$

and a similar argument takes care of the case where i is odd. Therefore,

$$\sum_{x\in R_i}|\sigma_x - H| \leqslant \frac{1}{4\beta}L\left[\frac{i}{L}\log(L/i) + C'\frac{i}{L}\right] \leqslant \frac{C''L}{\beta}$$

for some constants $C', C'' > 0$ independent of i, β, L.

Let us go back to estimating (6.11). It is clear that the x such that $d(x) \leqslant L/\log L$ can give altogether a contribution to $\sum_x [\sigma_x - \eta_x^{\sqcup}(T_1)]^+$ which is at most $O(L)$. Then, let $\tilde R_i$ be the subset of R_i such that $d(x) > L/\log L$. We can conclude that it is enough to estimate

$$\begin{aligned}&\mathbb{P}\left(\sum_{x\in\tilde R_i}[\sigma_x - \eta_x^{\sqcup}(T_1)]^+ \geqslant L(\ell(k+1,L) - C''')\right)\\ &\qquad\leqslant \mathbb{P}\left(\sum_{x\in\tilde R_i}[\sigma_x - \eta_x^{\sqcup}(T_1)]^+ \geqslant \frac{|\tilde R_i|}{2}\ell(k+1,L)\right).\end{aligned} \tag{6.15}$$

Now for every $x \in \tilde R_i$ define a (diagonal) interval $I_x \subset R_i$, centered at x and of length

$$|I_x| = \frac{1}{2}\min\left(d(x), \frac{L}{A\log^{(k)}(L)}\right). \tag{6.16}$$

Note that the minimal $|I_x|$ is of order $L/\log L$ and the maximal one is at most $L/(2A\log^{(k)}(L))$ (it can be much shorter if $|R_i| \ll L/\log^{(k)}(L)$). Note that condition (6.16) guarantees that around each I_x one can place a square Q_x of side $m_x = 2|I_x|$ and fully contained in Λ_L. Considering all the possible $i \leqslant L$ and the different intervals $I_x, x \in \tilde{R}_i$, the number of such intervals is trivially smaller than $|\Lambda_L|$. Therefore, observing that $\tilde{R}_i$ can be covered by (possibly overlapping) such intervals I_x of total length at most $(3/2)|\tilde{R}_i|$, it is enough to prove

$$\mathbb{P}\left(\sum_{y\in I_x}[\sigma_y - \eta_y^{\sqcup}(T_1)]^+ \geqslant \frac{|I_x|}{3}\ell(k+1,L)\right) \leqslant L^{-3} \tag{6.17}$$

for every such interval and then apply a union bound to get that the r.h.s. of (6.15) is $o(1/L)$, so that after summing over the index of R_i the probability in (6.11) is still $o(1)$.

It is easy but crucial to check that

$$\sum_{y\in I_x}[\sigma_y - \eta_y^{\sqcup}(T_1)]^+ \leqslant \sum_{y\in I_x}[H(m_x) - \eta_y^{\sqcup}(T_1)]^+ + \frac{c}{\beta}|I_x|(\log A + \log^{(k+1)}(L)) \tag{6.18}$$

where of course, as in (1.3), $H(m_x) = 1/(4\beta)\log m_x$ is just the typical equilibrium height of the SOS interface in the center of the square Q_x with zero boundary conditions on ∂Q_x (here, for lightness of notation, we forget the integer part in the definition of H). Indeed, since $m_x \leqslant d(x)/2$ one has for $y \in I_x$

$$|\sigma_y - H(m_x)| = \frac{1}{4\beta}(\log d(y) - \log(m_x)). \tag{6.19}$$

If $\min(d(x), L/A\log^{(k)}(L)) = d(x)$ then the r.h.s. of (6.19) is upper bounded by a constant. In the opposite case, it is bounded by

$$\frac{1}{4\beta}\left[\log L - \log\left(\frac{L}{A\log^{(k)}(L)}\right)\right] \leqslant \frac{1}{4\beta}(\log A + \log^{(k+1)}(L))$$

and (6.18) follows. Therefore, it is enough to bound

$$\mathbb{P}\left(\sum_{y\in I_x}[H(m_x) - \eta_y^{\sqcup}(T_1)]^+ \geqslant C_0|I_x|\right) \leqslant L^{-3} \tag{6.20}$$

for all such intervals, for some C_0 independent of β. We can assume that C_0 is large (just choose B large in (6.4)).

Monotonicity implies that if we let evolve *only the heights inside* Q_x with 0-b.c. on ∂Q_x, then the random configuration obtained at time T_1 is stochastically lower than the configuration obtained via the true evolution (where *all the heights* are updated). Again by monotonicity (the event in (6.20) being decreasing) we can lower the ceiling in the box Q_x from height $n^+ = \log L$ to height $\log m_x$ and also replace the pre-factor $(1/L)$ with $(1/m_x)$ in front of the fields $f_y, y \in Q_x$ in (6.1): the dynamics thus obtained (that we simply call "the auxiliary dynamics") gets stochastically lower. The reason is that the fields f_y are decreasing functions of η, which tend to "push down" the interface, and $1/m_x > 1/L$, so that $\exp((1/L - 1/m_x)f_y)$ is an increasing function.

Since we are assuming that $\mathcal{F}_k$ holds (with $a = 4$), the mixing time of the auxiliary dynamics in Q_x (with 0-b.c. on ∂Q_x) is at most

$$m_x^4 \exp(Am_x \log^{(k)}(m_x)) \leqslant L^4 \exp(L).$$

As a consequence, using (2.4), at time $T_1 = e^{2L}$ the law of the auxiliary dynamics is within variation distance $\exp(-e^{L/2})$ from its invariant measure, call it π_{Q_x}, which is nothing but a space translation of $\pi^{0,f}_{\Lambda_{m_x}}$, where we recall that, for a generic L, $\pi^{0,f}_{\Lambda_L}$ is the equilibrium measure in Λ_L with the field f, the floor/ceiling constraints $0 \leqslant \eta \leqslant \log L$ and b.c. at zero. For simplicity, for the rest of this subsection, we shift the square Λ_L so that its center coincides with the origin of $\mathbb{Z}^2$.

In conclusion,

$$\mathbb{P}\left(\sum_{y\in I_x}[H(m_x)-\eta^{\sqcup}_y(T_1)]^+ \geqslant C_0|I_x|\right) \leqslant e^{-e^{L/2}} + \pi^{0,f}_{\Lambda_{m_x}}\left(\sum_{y\in I}[H(m_x)-\eta_y]^+ \geqslant C_0|I|\right)$$

and I is a diagonal segment of cardinality $|I| = |I_x| = m_x/2$, centered at the origin of $\mathbb{Z}^2$. Thus we need the following equilibrium estimate:

Lemma 6.9. *For any m, if I is a diagonal segment of length $|I| = m/2$ centered at the origin of $\mathbb{Z}^2$, then:*

$$\pi^{0,f}_{\Lambda_m}(\mathcal{B}) := \pi^{0,f}_{\Lambda_m}\left(\sum_{y\in I}[H(m)-\eta_y]^+ \geqslant C_0|I|\right) \leqslant c\exp(-\beta m/c) \tag{6.21}$$

where $c > 0$ is a constant and Λ_m denotes the side-m square centered at the origin.

This will then be applied with m ranging from order $L/\log L$ to order $L/\log^{(k)}(L)$ so in all cases the r.h.s. is much smaller than L^{-3} and, putting everything together, the inequality (6.20) and therefore the claim of Lemma 6.5 follows. □

Proof of Lemma 6.9. Suppose this is true for the model without the field f, i.e. for the standard SOS measure $\pi^0_{\Lambda_m}$ of (2.2). Then, the same estimate follows (for β large, with c replaced by $c/2$) for $\pi^{0,f}_{\Lambda_m}$. This is so because, uniformly,

$$\frac{1}{m}\sum_{y\in\Lambda_m} f_y \leqslant c' m\, e^{-\beta/c'}$$

for some c' independent of β. To show that $\pi^0_{\Lambda_m}(\mathcal{B})$ is small, one first proves that

$$\hat{\pi}^{H(m)}_{\Lambda_m}(\mathcal{B}) \leqslant \exp(-(C_0/4)\beta m) \tag{6.22}$$

say for every $|I|$ of size between $\frac{1}{2}m$ and $\frac{2}{3}m$, where we recall from Section 2.1 that $\hat{\pi}^{H(m)}_{\Lambda_m}$ is the SOS measure without floor/ceiling and boundary conditions at height $H(m)$. This is based on Peierls-type arguments and the proof is relegated to Appendix D.

We conclude the proof of Lemma 6.9 assuming (6.22). Define $\Delta_i, i = 1,\dots,m/2-1$ to be the boundary of the square of side $m-2i$ centered at zero. Let E_i be the event

$$E_i = \Big\{\sum_{x\in\Delta_i}[H(m)-\eta_x]^+ \geqslant \delta m\Big\} \tag{6.23}$$

for some δ to be chosen small later. Suppose that at least one of the $E_i, i \leqslant m/10$ is not realized, and let j be the smallest such i. In that case, we look at the $\pi^0_{\Lambda_m}$-probability of $\mathcal{B}$, conditionally on the configuration of η on Δ_j. For all $x \in \Delta_j$, if $\eta_x > H(m)$ we can lower it to $H(m)$ by monotonicity (the event $\mathcal{B}$ is decreasing). If instead $\eta_x < H(m)$, we still change η_x by brute force to $H(m)$: the price to pay is that in the final estimate we get a multiplicative error

$$\exp\left(c\beta\sum_{x\in\Delta_j}[H(m)-\eta_x]^+\right) \leqslant e^{c\beta\delta m}$$

for some explicit c (independent of β and δ). What we get is that, conditionally on $j \leqslant m/10$ being the smallest index such that E_j is not realized, the $\pi^0_{\Lambda_m}$-probability of $\mathcal{B}$ is upper bounded by

$$e^{c\beta\delta m}\hat{\pi}^{H(m)}_{\Lambda_{m-2j}}(\mathcal{B}|0 \leqslant \eta \leqslant \log m) \leqslant e^{c\beta\delta m}\hat{\pi}^{H(m)}_{\Lambda_{m-2j}}(\mathcal{B}|\eta \leqslant \log m) \tag{6.24}$$

where the inequality is just monotonicity. Notice that $\hat{\pi}^{H(m)}_{\Lambda_{m-2j}}(\eta \leqslant \log m)$ is large (say, larger than $1/2$, cf. Proposition 3.9). Then, we can apply (6.22), since the interval I we are looking at is of length $m/2$, so that certainly $\frac{1}{2}(m-2j) \leqslant |I| \leqslant \frac{2}{3}(m-2j)$ and we get that the r.h.s. of (6.24) is upper bounded by

$$\exp(c\beta\delta m - (C_0/4)\beta(m-2j)).$$

At this point it is enough to choose δ small enough, for instance $\delta = C_0/(20c)$, to conclude (recall that $j \leqslant m/10$).

Next, we have to show that

$$\pi^0_{\Lambda_m}\left(\cap_{i=1}^{m/10} E_i\right) \tag{6.25}$$

is very small. Indeed, that event implies that

$$\sum_{x\in\Lambda_m}[H(m)-\eta_x]^+ \geqslant \delta m^2/10 = C_0 m^2/(200c). \tag{6.26}$$

Write

$$\sum_{x\in\Lambda_m}[H(m)-\eta_x]^+ = \sum_{k>0} kN_k \tag{6.27}$$

where N_k is the number of points where $[H(m)-\eta_x]^+ = k$. From Theorem 3.1 we know that there exists some integer K such that $N_k \leqslant m^2 e^{-2\beta k}$, except with probability $\exp(-m\exp(\beta k))$, for $k \geqslant K$. Then, except with probability of order $\exp(-c\beta m)$ one has $\sum_{k\geqslant 1} kN_k < C_0 m^2/(200c)$ if C_0 is chosen large enough (recall that, as discussed after (6.20), we can assume that C_0 is large). ☐

6.5. **Falling down from the ceiling: Proof of Lemma 6.6.** This is the part which requires the more subtle equilibrium estimates. Let $T_2 = \exp(c\beta L)$ where c will be determined along the proof. We want to prove that

$$\mathbb{P}(\eta^{\sqcap}(T_2) \in G^+_{\ell(\infty,L)}) > \tfrac{3}{4}. \tag{6.28}$$

We recall that $\ell(\infty, L) = B + 1/(4\beta)\log A$ is a constant that we can assume to be large. For simplicity, we write ℓ instead of $\ell(\infty, L)$.

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Ideally the proof would work as follows. At equilibrium, the event G^+_ℓ has probability almost 1, see Lemma 6.11 below (since G^+_ℓ is decreasing, in Lemma 6.11 we lift the boundary conditions on $\partial\Lambda_L$ from 0 to $H' = H+1$, the reason for the "+1" being that, in this way, for β large the floor has little influence on the interface at the typical height H'.) It is therefore sufficient to prove that at time T_2 the dynamics (with b.c. 0) is close to equilibrium. For this purpose we will apply Theorem 2.2 (which is allowed since we start from the maximal configuration $\sqcap$) with the following censoring schedule.

Cover Λ_L with overlapping, parallel rectangles V_i, $i \leqslant M = O(\log L)$, ordered from left to right, with longer vertical side L and shorter horizontal side

$(L/(\log L))$ and such that $V_i \cap V_{i+1}$ is a rectangle $L \times (L/(2\log L))$. Now consider the "bricks" B_i which have base V_i and height $n^+ = \log L$.

We first let B_1 evolve for a time $t_1 = \exp((c/2)\beta L)$. This is the SOS dynamics with b.c. 0 on the left, top and bottom boundary of V_1, and with b.c. n^+ on the right boundary. As we justify below, we can pretend that at time t_1, the system in B_1 has reached its own equilibrium. This equilibrium, restricted say to the left half of B_1, should be extremely close to the true equilibrium in Λ_L with 0 b.c. This can be justified as follows. The b.c. around V_1 impose the presence of open contours at heights $1, \ldots, n^+$, with endpoints at the endpoints of the right side of V_1. These contours behave roughly like random walks and will stay within distance say $L^{1/2+\varepsilon}$ from the right side of V_1 and only with tiny probability will intersect the left half of V_1.

Next, we let B_2 evolve for the same amount of time t_1, after which a similar argument shows that the "true equilibrium" is reached in the left half of V_2, i.e. on the right half of V_1, and so on. When the M^{th} block has been updated, the system should be very close to equilibrium everywhere. In practice, there are two major obstructions that prevent this strategy from being implemented directly and which cause much technical pain. The first has to do with the presence of the floor constraint at zero and will be discussed in greater detail in the Reader's Guide 6.13 below. The second difficulty can be understood in the following simplified situation.

Take the SOS in a $L \times m$ rectangle R, with $\sqrt{L} \ll m \ll L$ (for us, R would be V_1 so that $m = L/\log L$) with b.c. 1 on one of the size-L sides and b.c. 0 everywhere else, without any floor/ceiling. There is an open 1-contour joining the endpoints of the side with 1 b.c. The probability of such contour γ can be shown, via cluster expansion, to be proportional to $\exp(-\beta|\gamma| + \Psi_R(\gamma))$ where the "decoration" term $\Psi_R(\gamma)$ is of order $|\gamma|$ times a constant which is small with β. In absence of decorations, γ would behave as a random walk and it would be very unlikely that it reaches distance $\gg \sqrt{L}$ from the side with 1-b.c. In presence of the decorations, this might *in principle* fail. Indeed, the decorations depend also on how close the contour is to the boundary of R (see Appendix A), and this could induce a pinning effect of the contour on the size-L side with 0-b.c. The way out we found to exclude this scenario is a series of monotonicity arguments which in practice boil down to transforming R into a rectangle with *both* sides of order L. In this situation, since the side with 1-b.c. is very far from the opposite side, the "pinning effect" can be shown not to occur.

To prove (6.28) we couple $\eta^{\sqcap}(T_2)$ with a suitable equilibrium distribution as follows. Let Λ be the $2L \times L$ rectangle obtained by attaching a square of side L to the left of the original square Λ_L. Let $\pi_\Lambda^{H',f}$ denote the SOS equilibrium distribution in Λ with boundary conditions $H' := H + 1$. Such equilibrium measure contains the field f, cf. (6.1) (where the sum now is over $y \in \Lambda$ and the pre-factor is still $1/L$) and floor/ceiling constraints $0 \leqslant \eta \leqslant n^+$. One has

Lemma 6.11. *If ℓ is large enough, then*

$$\lim_{L\to\infty} \pi_\Lambda^{H',f}(\eta \in G_\ell^+) = 1. \tag{6.29}$$

The proof is deferred to Appendix D.

Therefore, using that the event G_ℓ^+ is decreasing, (6.28) follows if we prove that there exists a coupling of $(\eta, \eta^{\sqcap}(T_2))$, where η is the restriction to Λ_L of the configuration distributed according to $\pi_\Lambda^{H',f}$, such that

$$\mathbb{P}(\eta^\sqcap(T_2) \leqslant \eta) = 1 + o(1). \tag{6.30}$$

To this end, we will apply Theorem 2.2 with exactly the censoring described above. We first let evolve the system in B_1 for a time-lag t_1, with n^+ b.c. on the right side of V_1 and 0 b.c. elsewhere. Then we let evolve the system in B_2, for another time-lag t_1. For B_2 we have the maximal b.c. n^+ on the right boundary, zero b.c. on top and bottom and the b.c. on the left boundary is given by the configuration, say τ_1, inherited from the previous evolution on B_1. We repeat this procedure for the other bricks B_i, $i < M$, with maximal b.c. on the right boundary, zero b.c. on top and bottom and the b.c. τ_{i-1} on the left boundary; the final brick B_M, unlike the previous ones, has a zero b.c. on the right boundary as well as on the top and bottom boundaries, and b.c. τ_{M-1} on the left boundary.

We let $\widetilde{\eta}$ denote the configuration at the end of the above described procedure. Note that altogether the time spent is $Mt_1 \leqslant T_2 = \exp(c\beta L)$. Theorem 2.2 implies that we can couple $\widetilde{\eta}$ and $\eta^\sqcap(T_2)$ in such a way that $\mathbb{P}(\eta^\sqcap(T_2) \leqslant \widetilde{\eta}) = 1$. Thus, it remains to prove that (6.30) is satisfied with $\widetilde{\eta}$ replacing $\eta^\sqcap(T_2)$.

The mixing time of a brick is bounded above by $\exp((c/4)\beta L)$, for a suitable choice of $c > 0$, see Proposition 2.3. Therefore, after time t_1 the chain is extremely close to its equilibrium in B_1 with the given boundary conditions. Up to a global error term of order e^{-L} we can thus assume that after each updating of a brick, the corresponding random variable is given exactly by the equilibrium distribution on that brick with the prescribed boundary conditions (see Eq. (2.4)). Let $\widetilde{\eta}_i$ denote the configuration after the updating of brick B_i, restricted to the left half of the brick, i.e. the brick with basis $V_i' := V_i \cap (V_{i+1})^c$. Thus, using monotonicity, it is sufficient to exhibit a coupling such that

$$\mathbb{P}(\widetilde{\eta}_i \leqslant \eta_i,\ i = 1, \ldots, M-1) = 1 + o(1), \tag{6.31}$$

where η_i denotes the configuration η with distribution $\pi_\Lambda^{H',f}$, restricted to V_i'.

To prove the latter estimate we proceed as follows. Let $\mathcal{V}^i$ denote the portion of Λ_L covered by rectangles $V_1, \ldots, V_i$, and set $\mathcal{V}^0 := V_1'$. For $i = 0, \ldots, M$, call Λ^i the rectangle obtained by attaching a square of side L to the left of $\mathcal{V}^i$ (this corresponds to the "rectangle enlarging procedure" outlined above), and let ξ denote the b.c. equal to:

$$\xi_x = \begin{cases} n^+ & \text{if } x \text{ belongs to the right boundary of } \Lambda^i \\ H' & \text{otherwise.} \end{cases} \tag{6.32}$$

Since $V_i' \subset \Lambda^{i-1}$, by monotonicity and a repeated application of the DLR property for the measure $\pi_\Lambda^{H',f}$, we see that the desired claim (6.31) is a consequence of the next equilibrium result.

Theorem 6.12. *For every $C > 0$, there exists β_0 such that for all $\beta \geqslant \beta_0$, for all $i = 1, \ldots, M$,*

$$\|\pi_{\Lambda^i}^{\xi,f} - \pi_\Lambda^{H',f}\|_{\Lambda^{i-1}} \leqslant L^{-C}, \tag{6.33}$$

where $\|\cdot\|_{\Lambda^i}$ denotes total variation of the marginal on Λ^i.

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We now explain why (6.33) should be true and why we crucially need the field f, which is absent in the standard SOS measure (2.2). For simplicity, suppose that the boundary height ξ at the right vertical side of Λ^i is $H'+1$ instead of $n^+ = \log L$. There is an open $(H'+1)$-contour with endpoints at the endpoints of the side with b.c. $H'+1$. The probability that this contour equals γ should be approximately given by the product of three factors:

(i) the factor $\exp(-\beta(|\gamma|-L))$ (the minimal length of the open contour is L and one pays for the excess length);
(ii) a factor $\exp(+a(\beta)A(\gamma)/L)$ (with $A(\gamma)$ the area to the right of the contour); this is due to the entropic repulsion and $a(\beta)$ should be approximately $a(\beta) = \exp(-4\times 2\times\beta)$, where the factor 2 is due to the fact that $H'+1-H=2$;
(iii) $\exp(-b(\beta)A(\gamma)/L)$ where $b(\beta)$ is approximately given (for β large) by $b(\beta) = c_2(\beta) = \exp(-2\beta)$ which appears in (6.1).

Therefore, if β is large the third term beats the second one and one pays both excess length and excess area, and it should be very unlikely that the contour reaches distance $L/(\log L) \gg \sqrt{L}$ from the right rectangle side to which it is attached. We will find this probability to be roughly as small as $\exp(-cL/(\log L)^2)$, as would be the case for a random walk. Once we know the contour γ does not go much farther than $\sqrt{L}$ away from the side of the rectangle, a suitable coupling argument will prove the theorem; see Section 7.1. Remark that without the Hamiltonian modification (6.1) (i.e. with $f_y \equiv 0$) the area gain kills the length penalization, and the contour *would indeed* invade the rectangle Λ^i.

7. Proof of Theorem 6.12

The proof of Theorem 6.12 is based on the following lemma. Fix $i=1,\dots,M$ and set $R := \Lambda^i$, $R' := \Lambda^{i-1}$, so that the rectangle $R\setminus R'$ has horizontal length 2ℓ, where $\ell := L/(4\log L)$. Let also R'' denote the rectangle of points in R at distance at least ℓ from the right boundary. Note that $R \supset R'' \supset R'$, and $d(R\setminus R'', R') = \ell$, see Figure 3.

Let $\gamma_j(\eta)$, $j = H'+1,\dots,n^+$ denote the unique open j-contour in the rectangle R enforced by the boundary conditions, attached to the right boundary.

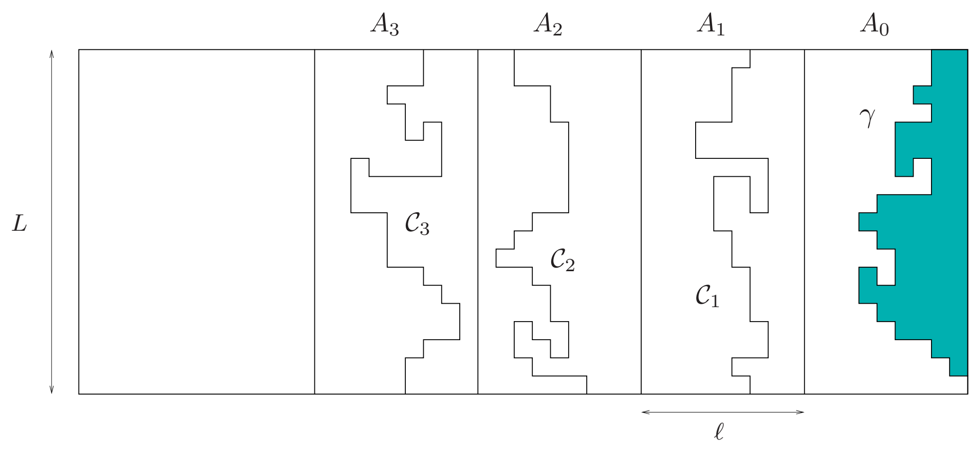

FIGURE 3. Schematic drawing of the rectangles R, $R'' = R\setminus A_0$, $R' = R''\setminus A_1$. Here the contour $\gamma = \gamma_{H'+1}$ illustrates the event $\mathcal{B}$ in Lemma 7.1, while the chains $\mathcal{C}_i$ in A_i illustrate the vertical crossings used in the proof of Theorem 6.12. The shaded region corresponds to $\mathrm{Int}(\gamma)$, while $\Lambda(\gamma) = R\setminus\mathrm{Int}(\gamma)$. The boundary between A_2 and A_1 is $\partial R'$.

Lemma 7.1. *Let $\mathcal{B}$ be the event that $\gamma_{H'+1}(\eta)$ does not intersect the rectangle R''. For every $C > 0$, $\Pi^{\xi,f}_{\Lambda^i}(\mathcal{B}^c) = O(L^{-C})$ where $\Pi^{\xi,f}_{\Lambda^i}$ is as in Notation 2.1 and ξ is as in (6.32).*

We first show how to obtain Theorem 6.12 from the estimate in Lemma 7.1. The proof of Lemma 7.1 is given in Section 7.2.

7.1. From Lemma 7.1 to Theorem 6.12.

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Let us first give a rough sketch of the coupling argument to be used. By conditioning on the value γ of the contour $\gamma_{H'+1}$ one can roughly replace the measure $\pi^{\xi,f}_{\Lambda^i}$ appearing in Theorem 6.12 by the measure $\Pi^{H',f}_{\Lambda(\gamma)}$, where $\Lambda(\gamma) := \Lambda \setminus \mathrm{Int}(\gamma)$ is the region to the left of γ. Strictly speaking this is not true but we shall reduce to a similar situation by way of monotonicity arguments. Also, thanks to the argument of Lemma 3.2, one can neglect the influence of the ceiling constraint. Thus, one essentially wants to couple $\Pi^{H',f}_{\Lambda(\gamma)}$ and $\Pi^{H',f}_{\Lambda}$ on the region $\Lambda^{i-1} = R'$. Thanks to Lemma 7.1, one can assume that the rectangle A_1 is contained in $\Lambda(\gamma)$. From the Markov property, it is sufficient to couple $\Pi^{H',f}_{\Lambda(\gamma)}$ and $\Pi^{H',f}_{\Lambda}$ on the interface separating the rectangles A_1 and A_2, see Figure 3. Thus, the desired estimate would follow if one could exhibit a coupling such that with large probability there exist chains $\mathcal{C}_1, \mathcal{C}_2$ of sites in the rectangles A_1, A_2 respectively, where both configurations are at constant height H'; see Figure 3. If there were no external fields and no wall constraint this would be a simple consequence of Lemma A.2 (recall that, for β large, the interface is rigid and there is a density close to 1 of sites where the height equals the boundary height). However, due to the presence of the external fields and the floor at zero, establishing this fact requires extra work. The idea here is to reduce the effective size of the system by imposing boundary conditions H' on vertical crossings in the rectangle A_3, and in the rectangle A_0. More precisely, let ρ, ρ_1 denote two vertical crossings in A_3, and let ρ_2 denote a vertical crossing in A_0. Using monotonicity and the estimate of Lemma C.1 of Appendix C, we replace $\Pi^{H',f}_{\Lambda(\gamma)}$ by $\Pi^{H',f}_{\Lambda(\rho,\gamma)}$ and $\Pi^{H',f}_{\Lambda}$ by $\Pi^{H',f}_{\Lambda(\rho_1,\rho_2)}$, where $\Lambda(\rho,\gamma)$ is the region between the chains ρ and γ, while $\Lambda(\rho_1,\rho_2)$ denotes the region between the chain ρ_1 and the chain ρ_2. Once this reduction has been achieved, the system is contained in the union of the four rectangles $\cup_{i=0}^4 A_i$, a $L \times 4\ell$ rectangle, and one can easily show that since ℓ is much smaller than L, and since $H' = H+1$, the external field and the wall constraint can be neglected; see the proof of Lemma 7.3 below. At this point, one can use Lemma A.2 to obtain the existence of chains $\mathcal{C}_1, \mathcal{C}_2$ with the properties mentioned above.

We turn to the details of the proof. It is sufficient to couple $\pi^{\xi,f}_{\Lambda^i}$ and $\pi^{H',f}_{\Lambda}$ on $\partial R'$, the set of points in R' with a nearest neighbor in $R \setminus R'$, i.e.

$$\|\pi^{\xi,f}_{\Lambda^i} - \pi^{H',f}_{\Lambda}\|_{R'} = \|\pi^{\xi,f}_{\Lambda^i} - \pi^{H',f}_{\Lambda}\|_{\partial R'}.$$

Note that, because $\pi^{\xi,f}_{\Lambda^i}$ has maximal b.c. n^+ on a side of R and b.c. coinciding with that of $\pi^{H',f}_{\Lambda}$ on the other sides, $\pi^{\xi,f}_{\Lambda^i}$ stochastically dominates $\pi^{H',f}_{\Lambda}$ on $\partial R'$ and therefore, by a union bound, one has

$$\|\pi^{\xi,f}_{\Lambda^i} - \pi^{H',f}_{\Lambda}\|_{R'} \leqslant \sum_{x\in\partial R'} \sum_{v=0}^{n^+-1} \left[\pi^{\xi,f}_{\Lambda^i}(U_{x,v}) - \pi^{H',f}_{\Lambda}(U_{x,v})\right], \tag{7.1}$$

where we define the events $U_{x,v} := \{\eta_x > v\}$. Next, we remove the ceiling constraint from the measures $\pi^{\xi,f}_{\Lambda^i}, \pi^{H',f}_{\Lambda}$. Since $U_{x,v}$ are monotone events, we can estimate $\pi^{\xi,f}_{\Lambda^i}(U_{x,v}) \leqslant \Pi^{\xi,f}_{\Lambda^i}(U_{x,v})$. Moreover, as in Section 6.3, one has $\pi^{H',f}_{\Lambda}(U_{x,v}) = \Pi^{H',f}_{\Lambda}(U_{x,v}) + O(L^{-C})$ where C is as large as we wish provided β is sufficiently large.

Let[3] $\tilde{\mu}^\gamma$ denote the marginal on R'' of $\Pi^{\xi,f}_{\Lambda^i}$ conditioned to have $\gamma_{H'+1}(\eta) = \gamma$. Let $\mathrm{Int}(\gamma)$ denote all sites enclosed by the contour γ and the right boundary of R, cf. Figure 3. Let Δ^-_γ denote the set of sites $x \in R \setminus \mathrm{Int}(\gamma)$ that have either a nearest neighbor in $\mathrm{Int}(\gamma)$, or a site at distance $\sqrt{2}$ in $\mathrm{Int}(\gamma)$ in either the south-west or north-east direction. Since conditioning on $\gamma_{H'+1}(\eta) = \gamma$ forces all sites in Δ^-_γ to be at height $\eta_x \leqslant H'$ (recall Definition 3.3 of an h-contour), by monotonicity one has $\tilde{\mu}^\gamma(U_{x,v}) \leqslant \mu^\gamma(U_{x,v})$ if μ^γ denotes the marginal on R'' of $\Pi^{\xi,f}_{\Lambda^i}$ conditioned to have height exactly H' on all sites $x \in \Delta^-_\gamma$. Writing $\mathcal{B}$ for the event that $\gamma_{H'+1}(\eta)$ does not intersect the rectangle R'', one has, uniformly in x, v:

$$\pi^{\xi,f}_{\Lambda^i}(U_{x,v}) - \pi^{H',f}_{\Lambda}(U_{x,v}) \leqslant L^{-C} + \Pi^{\xi,f}_{\Lambda^i}(\mathcal{B}^c) + \max_{\gamma\in\mathcal{B}} \mu^\gamma(U_{x,v}) - \Pi^{H',f}_{\Lambda}(U_{x,v}). \tag{7.2}$$

Lemma 7.1 says that $\Pi^{\xi,f}_{\Lambda^i}(\mathcal{B}^c) = O(L^{-C})$, so that we are left with the upper bound on $\mu^\gamma(U_{x,v}) - \Pi^{H',f}_{\Lambda}(U_{x,v})$ for $\gamma \in \mathcal{B}$. We now implement the system reduction mentioned in the sketch of the proof above.

Let A_i, $i = 0, 1, 2, 3$, denote the $L \times \ell$ rectangles in R depicted in Figure 3. Write A^t_i, for the external top boundary of A_i, i.e. the set of sites $x \notin R$ such that x has a nearest neighbor on the top side of the rectangle A_i. Similarly, write A^b_i for the external bottom boundary of A_i. Call $\mathcal{E}(A_i)$ the set of $\mathbb{Z}^2$-bonds e such that e has at least one endpoint in A_i and at most one endpoint in $A^t_i \cup A^b_i$. A vertical crossing in A_i is a connected set $\mathcal{C} \subset \mathcal{E}(A_i)$ that connects A^t_i and A^b_i; see Figure 3.

Let $\mathcal{F}_-$ (resp. $\mathcal{F}_+$) denote the event that there exists a vertical crossing $\mathcal{C}$ in A_3 such that $\eta_x \leqslant H'$ for all $x \in \mathcal{C}$ (resp. a crossing $\mathcal{C}$ in A_3 and a crossing $\mathcal{C}'$ in A_0 such that $\eta_x \geqslant H'$, $x \in \mathcal{C} \cup \mathcal{C}'$). On the event $\mathcal{F}_-$ one may consider the leftmost vertical crossing in A_3 with the required property, where leftmost is defined according to lexicographic order. From the Markov property of μ^γ, and using monotonicity,

$$\mu^\gamma(U_{x,v}) \leqslant \mu^\gamma(\mathcal{F}^c_-) + \max_\rho \mu^{\gamma,\rho}(U_{x,v}),$$

where $\mu^{\gamma,\rho}$ stands for the measure μ^γ conditioned to have height H' on ρ, and ρ ranges over all possible vertical crossings in A_3. Similarly, on the event $\mathcal{F}_+$ denote ρ_1 (resp. ρ_2) the rightmost (resp. leftmost) crossing in A_0 (resp. A_3) and write

$$\Pi^{H',f}_{\Lambda}(U_{x,v}) \geqslant (1 - \Pi^{H',f}_{\Lambda}(\mathcal{F}^c_+)) \min_{\rho_1,\rho_2} \mathcal{Q}^{\rho_1,\rho_2}(U_{x,v}) \geqslant \min_{\rho_1,\rho_2} \mathcal{Q}^{\rho_1,\rho_2}(U_{x,v}) - \Pi^{H',f}_{\Lambda}(\mathcal{F}^c_+) \tag{7.3}$$

where $\mathcal{Q}^{\rho_1,\rho_2}$ stands for the measure $\Pi^{H',f}_{\Lambda}$ conditioned to have height H' on ρ_1, ρ_2, and ρ_1, ρ_2 range over all possible vertical crossings in A_0, A_3. Altogether,

$$\mu^\gamma(U_{x,v}) - \Pi^{H',f}_{\Lambda}(U_{x,v}) \leqslant \mu^\gamma(\mathcal{F}^c_-) + \Pi^{H',f}_{\Lambda}(\mathcal{F}^c_-) + \max_{\rho,\rho_1,\rho_2} \left|\mu^{\gamma,\rho}(U_{x,v}) - \mathcal{Q}^{\rho_1,\rho_2}(U_{x,v})\right|. \tag{7.4}$$

It follows from Lemma C.1 of Appendix C that $\mu^\gamma(\mathcal{F}^c_-)$ and $\Pi^{H',f}_{\Lambda}(\mathcal{F}^c_+)$ are $O(e^{-L^{1-\varepsilon}})$. Notice that $\mu^{\gamma,\rho}$ (resp. $\mathcal{Q}^{\rho_1,\rho_2}$) are SOS measures with exactly H' b.c. around the domain whose boundary is determined by ρ (resp. ρ_1) on the left and by Δ^-_γ (resp. ρ_2) on the right. Such

[3]In the sequel of the proof we introduce local notation for various conditional marginals of the measures $\Pi^{\xi,f}_{\Lambda^i}$, $\Pi^{H',f}_{\Lambda}$ in order to keep formulas readable.

domain has (by construction) horizontal size of order ℓ and vertical size L. To simplify the notation, we shall write $\mu^\gamma, \mathcal{Q}$ for $\mu^{\gamma,\rho}, \mathcal{Q}^{\rho_1,\rho_2}$.

We now turn our attention to vertical crossings in the rectangles A_1, A_2. Consider the independent coupling $\mathbb{P}$ of $\mu^\gamma, \mathcal{Q}$ on $A_1 \cup A_2$. Writing (η, η') for the corresponding random variables, let $\mathcal{A}_i$, $i = 1, 2$ denote the event that there exists a vertical crossing $\mathcal{C}$ in A_i such that $\nabla_e \eta = \nabla_e \eta' = 0$ for all bonds e with both endpoints in $\mathcal{C}$. Note that if $\mathcal{C}$ is a vertical crossing in A_i as above, then $\eta_\mathcal{C} = \eta'_\mathcal{C} = H'$, because of the boundary conditions equal to H' on the top and bottom boundary of A_i, $i = 1, 2$. On the event $\mathcal{A}_1 \cap \mathcal{A}_2$, one may consider the leftmost vertical crossing $\mathcal{C}_2$ in A_2 and the rightmost vertical crossing $\mathcal{C}_1$ in A_1. From the Markov property of the Gibbs measures $\mu^\gamma, \mathcal{Q}$ and the fact that $\mathcal{Q}(\cdot|\eta_{\mathcal{C}_1} = \eta_{\mathcal{C}_2} = H')$ and $\mu^\gamma(\cdot|\eta_{\mathcal{C}_1} = \eta_{\mathcal{C}_2} = H')$ have the same marginal on $\partial R'$ (observe that $\partial R'$ is just at the boundary between A_1 and A_2), one obtains that

$$|\mu^\gamma(U_{x,v}) - \mathcal{Q}(U_{x,v})| \leqslant \mathbb{P}(\mathcal{A}_1^c) + \mathbb{P}(\mathcal{A}_2^c). \tag{7.5}$$

We shall focus on the event $\mathcal{A}_1^c$, since the event $\mathcal{A}_2^c$ can be treated in the same way. To estimate $\mathbb{P}(\mathcal{A}_1^c)$, we use the fact (see e.g. [34, Lemma 11.21]) that non-existence of a vertical crossing in A_1 implies the existence of a horizontal dual crossing in A_1. More precisely, let A_1^r denote the right side of A_1, i.e. the set of dual bonds e' such that e' crosses an edge of the form $e = (x, y)$ with $x \in A_1$ and $y \in R \setminus R''$. Similarly, let A_1^ℓ denote the left side of A_1. We say that a dual bond e' is in A_1 if e' crosses a bond $e \in \mathcal{E}(A_1)$. Then, the event $\mathcal{A}_1^c$ implies that there exists a connected set $\mathcal{D}$ of dual bonds e' in A_1 which connects the lines A_1^r and A_1^ℓ, and such that for every $e' \in \mathcal{D}$ either $\nabla_{e'}\eta \neq 0$ or $\nabla_{e'}\eta' \neq 0$. Here we use the notation $\nabla_{e'}\eta := \nabla_e \eta$ if e' is the dual bond that crosses e. Moreover, for a given $\mathcal{D}$ as above, there must be a set $V \subset \mathcal{D}$ such that $|V| \geqslant |\mathcal{D}|/2$ and such that either $E_V := \{\nabla_{e'}\eta \neq 0 \text{ for all } e' \in V\}$ or $F_V := \{\nabla_{e'}\eta' \neq 0 \text{ for all } e' \in V\}$. Thus, using a union bound, one obtains

$$\mathbb{P}(\mathcal{A}_1^c) \leqslant \sum_{\mathcal{D}} \sum_{\substack{V \subset \mathcal{D}: \\ |V| \geqslant |\mathcal{D}|/2}} \left(\mu^\gamma(E_V) + \mathcal{Q}(F_V)\right) \tag{7.6}$$

where the first sum is over all connected sets of dual bonds $\mathcal{D}$ connecting A_1^r and A_1^ℓ as above. We will need the following lemma.

Lemma 7.3. *There exist constants $C, c, \beta_0 > 0$ independent of β such that, for every set V of dual bonds in A with $|V| \geqslant \ell/2$, one has for all $\beta \geqslant \beta_0$*

$$\max\{\mu^\gamma(E_V), \mathcal{Q}(F_V)\} \leqslant C e^{-c\,\beta|V|}. \tag{7.7}$$

Let us conclude the proof of Theorem 6.12 assuming for a moment the validity of Lemma 7.3. From (7.6), summing over the possible (connected) sets $\mathcal{D}$, and using $|\mathcal{D}| \geqslant \ell \geqslant L^{1-\varepsilon}$, for all $\varepsilon > 0$, if $\beta \geqslant \beta_0$:

$$\begin{aligned}
\mathbb{P}(\mathcal{A}_1^c) &\leqslant 2C \sum_{k \geqslant \ell} \sum_{\mathcal{D}:\,|\mathcal{D}|=k} \sum_{\substack{V \subset \mathcal{D}: \\ |V| \geqslant k/2}} e^{-c\beta|V|} \\
&\leqslant 2C \sum_{k \geqslant \ell} \sum_{\mathcal{D}:\,|\mathcal{D}|=k} 2^k e^{-c\beta k/2} \leqslant 2C \sum_{k \geqslant \ell} 6^k e^{-c\beta k/2} \\
&\leqslant C' e^{-c\beta\ell/4} = O(\exp(-L^{1-\varepsilon})).
\end{aligned} \tag{7.8}$$

Since the constants implied in (7.8) are uniform in x, v and the choice of $\gamma \in \mathcal{B}$, the claim of Theorem 6.12 follows from (7.2) and (7.1). It remains to prove Lemma 7.3. This is where the reduction from μ^γ to $\mu^{\gamma,\rho}$ and $\Pi_\Lambda^{H',f}$ to $\mathcal{Q}^{\rho_1,\rho_2}$ becomes important.

Proof of Lemma 7.3. We shall prove the bound concerning $\mu^\gamma = \mu^{\gamma,\rho}$ only, since the same proof works for $\mathcal{Q} = \mathcal{Q}^{\rho_1,\rho_2}$. Consider the region $\Lambda_0 \subset R$ delimited on the left by ρ and on the right by γ. Since ρ is a vertical crossing in A_3, one has $A_1 \subset \Lambda_0$. A crucial fact is that $|\Lambda_0| \leqslant 4L\ell$. Let as usual $\hat\pi^{H'}_{\Lambda_0}$ denote the SOS measure on Λ_0 with boundary condition H' outside of Λ_0, with no floor, no ceiling and no external fields. From Lemma A.2 one has $\hat\pi^{H'}_{\Lambda_0}(E_V) \leqslant e^{-\beta|V|/2}$ for any V. Thus, it suffices to show that

$$\mu^\gamma(E_V) \leqslant Ce^{C\ell}\,\hat\pi^{H'}_{\Lambda_0}(E_V), \tag{7.9}$$

for some constant C independent of β. Note that the external fields contribute with the term $0 \leqslant \frac{1}{L}\sum_{x\in\Lambda_0} f_x \leqslant C\ell$ to the Hamiltonian, and therefore, at the price of a factor $e^{C\ell}$ we can remove all external fields in our measure μ^γ. Then

$$\mu^\gamma(E_V) \leqslant e^{C\ell}\frac{\hat\pi^{H'}_{\Lambda_0}(E_V)}{\hat\pi^{H'}_{\Lambda_0}(\eta_x \geqslant 0,\,\forall x\in\Lambda_0)}\,. \tag{7.10}$$

Next, from the FKG inequality, one has

$$\hat\pi^{H'}_{\Lambda_0}(\eta_x \geqslant 0,\,\forall x\in\Lambda_0) \geqslant \prod_{x\in\Lambda_0}\hat\pi^{H'}_{\Lambda_0}(\eta_x\geqslant 0) \geqslant \prod_{x\in\Lambda_0}(1 - C\,e^{-4\beta H'})\,, \tag{7.11}$$

where we use the equilibrium estimate $\hat\pi^{H'}_{\Lambda_0}(\eta_x < 0) = \hat\pi^{0}_{\Lambda_0}(\eta_x > H') \leqslant Ce^{-4\beta H'}$; see Proposition 3.9. Since $e^{-4\beta H'} = e^{-8\beta}/L$, one has

$$\prod_{x\in\Lambda_0}(1 - C\,e^{-4\beta H'}) \geqslant C_1^{-1}e^{-C_1\ell}, \tag{7.12}$$

for a suitable constant $C_1 > 0$. The desired conclusion follows from (7.10). This ends the proof. □

7.2. Proof of Lemma 7.1.

Reader's Guide 7.4.

Roughly speaking, the proof of Lemma 7.1 works as follows. There are $n^+ - H'$ open contours attached to the right side of R (call it r) and let γ_j, $j \in \{H'+1,\dots,n^+\}$ denote the j-contour. First one proves that the n^+-contour cannot reach distance say $L/(\log L)^2$ from r. For this, one lifts from H' to n^+-1 the b.c. around the three sides of R different from r (this is allowed by monotonicity). This way, there is now *a single open contour* and the estimate follows from Proposition B.1. Next, we want to prove that γ_{n^+-1} cannot reach distance $L/(\log L)^2$ from γ_{n^+}, i.e. distance $2L/(\log L)^2$ from r. Morally the proof works as for the previous case, except that now the b.c. n^+ at r is replaced by the b.c. n^+-1 at γ_{n^+}. The argument is then repeated iteratively and the statement of the lemma follows when $j = H'+1$. In practice, there are many additional difficulties, which is why the proof is so much involved. The main obstacles are the following:

1. Proposition B.1 cannot be applied directly, because it holds when both the floor constraint $\eta \geqslant 0$ and the field f are absent. However, Proposition 7.7 will show that (morally) the field compensates the effect of the floor (which would tend to push the contours away from r);
2. Once γ_j is fixed, it is not true that the next contour (i.e. γ_{j-1}) sees boundary conditions $j-1$ in a new domain determined by γ_j. The point is that, from definition of contours, we only know that the heights just to its left are *at*

most $j-1$, not *exactly* $j-1$. We will use monotonicity to be able to change to $j-1$-b.c.

(3) Applying Proposition B.1 as outlined above to estimate the probability of large deviations of γ_{j-1} given γ_j requires that the right boundary of the system (that is the configuration of γ_j), where b.c. are $j-1$, is not too wild. In practice, one needs it to be a path connecting top and bottom of the rectangle R, with transversal fluctuations at most of order say L^ε for some small ε. We will apply the results of Appendix C to infer that, indeed, to the left of γ_j and not far away from it there is a chain of sites, with transversal fluctuations of the required order, where heights are exactly $j-1$.

We use a sort of induction on the index of the open contours γ_j, $j \in \{H'+1, \dots, n^+\}$, where $n^+ = \log L$. Let A_0 denote the rightmost $L \times \ell$ rectangle inside R as in Figure 3, and write $A_0 = \cup_{j=H'+1}^{n^+} B_j$ where B_j are non-overlapping $L \times \ell_0$ rectangles, ordered from left to right, such that $\ell_0 = \ell/(n^+ - H') \approx L/(4(\log L)^2)$. Every rectangle B_j is further divided into two non-overlapping rectangles B_j^1, B_j^2, ordered from left to right, such that B_j^1 is a $L \times \ell_1$ rectangle with $\ell_1 = L^\delta$, for some (arbitrarily) small $\delta > 0$, and $B_j^2 = B_j \setminus B_j^1$ is a $L \times \ell_2$ rectangle, with $\ell_2 = \ell_0 - \ell_1 \sim \ell_0$; see Figure 4.

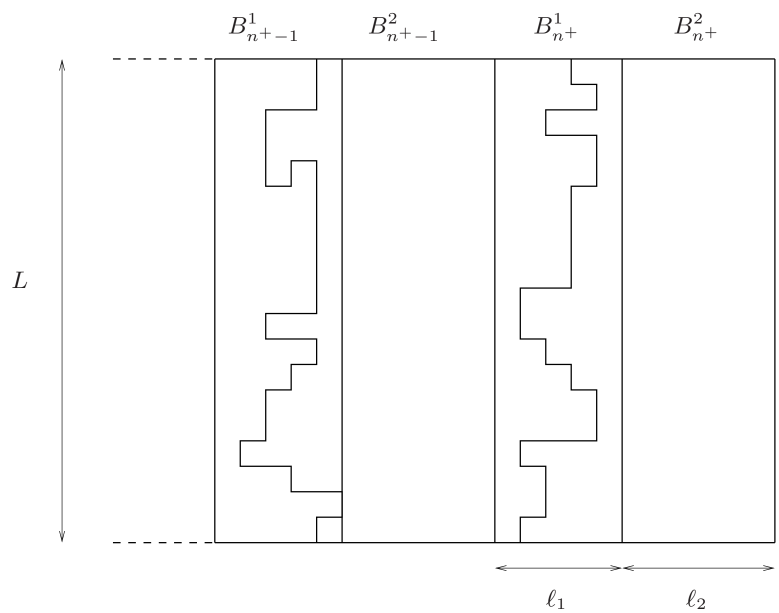

FIGURE 4. The rectangles B_j^1, B_j^2, with the associated vertical crossings ρ_j, for $j = n^+ - 1$ and $j = n^+$.

Define vertical crossings in a rectangle as in Section 7.1. For $j \in \{H'+1, \dots, n^+\}$, consider the event $\mathcal{B}_j$ that there exists a vertical crossing $\mathcal{C}_j$ in B_j^1, such that $\eta_x \leqslant j-1$ for all $x \in \mathcal{C}_j$. In particular, on $\mathcal{B}_{H'+1}$, there exists a vertical crossing $\mathcal{C}_{H'+1}$ in A_0 with $\eta_x \leqslant H'$ for all $x \in \mathcal{C}_{H'+1}$. Thus $\mathcal{B} \supset \mathcal{B}_{H'+1}$, and it will be sufficient to estimate from above the probability $\Pi_{\Lambda^i}^{\xi,f}(\mathcal{B}_{H'+1}^c)$. Clearly,

$$\Pi_{\Lambda^i}^{\xi,f}(\mathcal{B}_{H'+1}^c) \leqslant \sum_{j=H'+1}^{n^+} \Pi_{\Lambda^i}^{\xi,f}(\mathcal{B}_j^c \cap \mathcal{B}_{j+1}),$$

where $\mathcal{B}_{n^++1}$ denotes the whole probability space. On the event $\mathcal{B}_{j+1}$, let $\mathcal{C}_{j+1}$ denote the rightmost vertical crossing $\mathcal{C}$ in B^1_{j+1} such that $\eta_x \leqslant j$, $x \in \mathcal{C}$. By conditioning on the event $\{\mathcal{C}_{j+1} = \rho_{j+1}\}$, and using that the events $\mathcal{B}^c_j$ are increasing, one has

$$\Pi^{\xi,f}_{\Lambda^i}(\mathcal{B}^c_{H'+1}) \leqslant \sum_{j=H'+1}^{n^+} \max_{\rho_{j+1}} \mu_{\rho_{j+1}}(\mathcal{B}^c_j), \tag{7.13}$$

where ρ_{j+1} ranges over all possible vertical crossings in B^1_{j+1} (for $j = n^+$, it is understood that ρ_{j+1} coincides with the right boundary of R), and $\mu_{\rho_{j+1}}$ stands for the SOS Gibbs measure on the region $\Lambda_{(j)} \subset R$ defined as the set of sites $x \in R$ to the left of the crossing ρ_{j+1}, with

- boundary condition $\eta_x = j$ for $x \in \rho_{j+1}$ and $\eta_x = j-1$ on all other boundary sites. Note that a portion of the boundary height has been lifted from H' to $j-1 \geqslant H'$. The advantage is that, this way, there is a unique open contour under the measure $\mu_{\rho_{j+1}}$, rather than $j - H'$ of them ;
- floor constraint $\eta_x \geqslant 0$;
- external field

$$\frac{1}{L}\sum_{x\in\Lambda_{(j)}} f_{x,j-1-H}. \tag{7.14}$$

where we recall that $f_{x,j} = \exp(-\beta j)\mathbf{1}_{\eta_x \leqslant H+j}$, cf. (6.1). Note that the fields in (6.1) with index different from $j-1-H$ have been removed. This is allowed since the function $\eta \mapsto f_{x,a}(\eta)$ is decreasing. The effect of the field $f_{x,a}$ is to depress the area of the $(a+H+1)$-open contour and, since there is just one open contour, we need only the term with $a = j-1-H$.

Lemma 7.1 is then a consequence of (7.13) and the following claim.

Claim 7.5. *For $j \geqslant H'+1$, uniformly in the vertical crossing ρ_{j+1} in B^1_{j+1} and for every $C > 0$,*

$$\mu_{\rho_{j+1}}(\mathcal{B}^c_j) = O(L^{-C}). \tag{7.15}$$

Let γ_j denote the unique open j-contour for a configuration η in the ensemble $\mu_{\rho_{j+1}}$. By construction, γ_j is to the left of B^2_{j+1}, and it may intersect the rectangle B^2_j or even B^1_j. Let E_j denote the event that γ_j intersects B^1_j. Conditionally on the event E^c_j, the contour stays to the right of B^1_j, and the estimate $\mu_{\rho_{j+1}}(\mathcal{B}^c_j|E^c_j) = O(\exp(-L^\delta))$ follows from (C.2), which is applicable since the shorter side of R is at most of length L. Thus the claim (and hence Lemma 7.1) follows once we prove the following lemma.

Lemma 7.6. *For $j \geqslant H'+1$, uniformly in the vertical crossing ρ_{j+1} in B^1_{j+1}, and for all $C > 0$:*

$$\mu_{\rho_{j+1}}(E_j) = O(L^{-C}). \tag{7.16}$$

We will actually give an upper bound of order $\exp(-L^{1-\varepsilon})$ for every $\varepsilon > 0$.

Proof of Lemma 7.6. For this proof, the crossing ρ_{j+1} in B^1_{j+1} is fixed, and we simply write μ instead of $\mu_{\rho_{j+1}}$. Fix a contour Γ and consider the event $\gamma_j = \Gamma$. Set $\Lambda_+ = \mathrm{Int}(\Gamma) \cap \Lambda_{(j)}$, and $\Lambda_- = \Lambda_{(j)} \setminus \Lambda_+$, so that Γ is the set of dual bonds separating Λ_- and Λ_+ within $\Lambda_{(j)}$ (with $\mathrm{Int}(\Gamma)$ defined a few lines after (7.1)). For any Γ one may write

$$\mu(\gamma_j = \Gamma) \propto e^{-\beta|\Gamma|} Z_{j,\Lambda_-} Z_{j,\Lambda_+}. \tag{7.17}$$

Here, Z_{j,Λ_-} (resp. Z_{j,Λ_+}) is the partition function of the SOS model on Λ_- (resp. Λ_+), with floor at height 0, field as in (7.14), b.c. $j-1$ on $\partial\Lambda_-$ (resp. b.c. j on $\partial\Lambda_+$) and with the extra constraint that $\eta_x \leqslant j-1$ for all $x \in \Delta_\Gamma^-$ (resp. $\eta_x \geqslant j$ for all $x \in \Delta_\Gamma^+$), where Δ_Γ^- (resp. Δ_Γ^+) is the set of $x \in \Lambda_-$ either at distance 1 from Λ_+ (resp. Λ_-) or at distance $\sqrt{2}$ from a vertex $y \in \Lambda_+$ (resp. $y \in \Lambda_-$) in the south west or north east direction. These constraints are imposed by the definition of j-contour; see Definition 3.3.

Next, let $\mathcal{Z}^0_{\Lambda_-}$ (resp. $\mathcal{Z}^0_{\Lambda_+}$) denote the partition function of the SOS model on Λ_- (resp. Λ_+) with b.c. 0, no floor and no external fields, with the constraint that $\eta_x \leqslant 0$ for all $x \in \Delta_\Gamma^-$ (resp. $\eta_x \geqslant 0$, $x \in \Delta_\Gamma^+$). Let ω_-, ω_+ be the corresponding Gibbs measures. With these definitions, one rewrites (7.17) as

$$\mu(\gamma_j = \Gamma) = \frac{1}{\mathcal{Z}} e^{-\beta|\Gamma|} \mathcal{Z}^0_{\Lambda_-} \mathcal{Z}^0_{\Lambda_+} \times \\ \times\ \omega_-\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_-} \mathbf{1}_{\eta_x \leqslant 0}}\,;\, \eta \geqslant -(j-1)\Big)\ \omega_+\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_+} \mathbf{1}_{\eta_x \leqslant -1}}\,;\, \eta \geqslant -j\Big) \tag{7.18}$$

where $\mathcal{Z}$ is the normalization and $\mathcal{K} = c_{j-1-H} = \exp(-\beta(j-1-H))$, see (6.1).

We first observe that the very same arguments of Proposition 3.9 proves that for all $x \in \Lambda_\pm$,

$$\omega_\pm(\eta_x \geqslant j) \asymp e^{-4\beta j}, \tag{7.19}$$

that is $C^{-1}e^{-4\beta j} \leqslant \omega_\pm(\eta_x \geqslant j) \leqslant C\,e^{-4\beta j}$ for some absolute constant $C > 0$. This is possible thanks to the fact that even in the presence of the constraints on $\Delta_\Gamma^\pm$ the arguments of Lemma 3.7 can be used without modifications. Next, let $\hat\pi$ stand for the infinite volume limit of the SOS measure with zero boundary condition. Proposition 3.9 implies that $\hat\pi(\eta_0 \geqslant j) \asymp e^{-4\beta j}$. Moreover, we observe that there exist constants $c, t_0 > 0$ such that for any $x \in \Lambda_\pm$ at distance at least $t > t_0$ from the boundary $\partial\Lambda_\pm$, for any k:

$$|\omega_\pm(\eta_x \geqslant k) - \hat\pi(\eta_0 \geqslant k)| \leqslant e^{-c\,t}. \tag{7.20}$$

Let us prove (7.20) in the case $x \in \Lambda_-$. The case $x \in \Lambda_+$ is obtained with the same argument. Thanks to the exponential decay of correlations for the zero-b.c. SOS model at large β (see [11]), (7.20) is equivalent to the statement obtained by replacing $\hat\pi$ by $\hat\pi^0_{\Lambda_-}$. Observe that by monotonicity $\omega_-(\eta_x \geqslant k) \leqslant \hat\pi^0_{\Lambda_-}(\eta_x \geqslant k)$. Next, by the same argument of Lemma 3.7, the ω_--probability of a contour γ, is bounded above by $e^{-\beta|\gamma|}$, and thus with probability at most $e^{-c\,t}$ there is no chain $\mathcal{C}$ of heights all greater or equal to zero in a shell of width $t/2$ around x, and at distance larger than $t/2$ from x. On the other hand, if E is the event that such a chain exists then by monotonicity and decay of correlations one has $\omega_-(\eta_x \geqslant k; E) \geqslant \hat\pi^0_{\Lambda_-}(\eta_x \geqslant k) + e^{-c\,t}$. This proves (7.20).

We turn to a rough estimate that allows one to rule out very long contours. Namely, if G denotes the event that $|\gamma_j| \leqslant L^{1+\varepsilon_0}$, then for all β large enough

$$\mu(G) = 1 - O\big(e^{-L^{1+\varepsilon_0}}\big). \tag{7.21}$$

In what follows we may fix $\varepsilon_0 > 0$ as small as we wish. To prove (7.21), observe that from a trivial bound on the external fields and the FKG property for $\omega_\pm$ one has

$$\omega_-\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_-} \mathbf{1}_{\eta_x \leqslant 0}}\,;\, \eta \geqslant -(j-1)\Big) \geqslant \prod_{x\in\Lambda_-} \omega_-(\eta_x \geqslant -(j-1)). \tag{7.22}$$

From (7.19)

$$\omega_-(\eta_x \geqslant -(j-1)) \geqslant (1 - ce^{-4\beta(j-1)}) \geqslant \exp(-c'/L),$$

since $j \geqslant H$. Then (7.22) is bounded below by e^{-CL} for some $C > 0$. The same estimate holds for the last term in (7.18), and therefore one has

$$\mu(\gamma_j = \Gamma) \leqslant \nu(\Gamma) e^{CL}, \tag{7.23}$$

for some constant $C > 0$, where ν is the probability measure on contours Γ given by

$$\nu(\Gamma) \propto e^{-\beta|\Gamma|} \mathcal{Z}^0_{\Lambda_-} \mathcal{Z}^0_{\Lambda_+}. \tag{7.24}$$

Notice that ν is the distribution of the unique open contour of the SOS measure on $\Lambda_- \cup \Lambda_+$ with no floor constraint, with Dobrushin boundary conditions, namely with b.c. $\eta_y = 0$ or $\eta_y = 1$ depending on whether y has a nearest neighbor in Λ_- or in Λ_+ respectively. It follows from (B.1) that $\nu(\Gamma)$ has the standard form

$$\nu(\Gamma) \propto e^{-\beta|\Gamma| + \Psi(\Gamma)}$$

where the decoration term Ψ satisfies $|\Psi(\Gamma)| \leqslant c e^{-\beta}|\Gamma|$. The usual Peierls' argument shows that

$$\nu(|\gamma| \geqslant L^{1+\varepsilon_0}) = O(e^{-L^{1+\varepsilon_0}}), \tag{7.25}$$

and (7.21) follows.

Thanks to (7.21), we can now restrict the summation in the normalization $\mathcal{Z}$ in (7.18) to contours $\Gamma \in G$. Define

$$\Phi_- := \frac{\mathcal{K}}{L}|\Lambda_-|\,\hat\pi(\eta_0 > 0)\,, \quad \Psi_- := |\Lambda_-|\,\hat\pi(\eta_0 < -(j-1))\,, \tag{7.26}$$

$$\Phi_+ := \frac{\mathcal{K}}{L}|\Lambda_+|\,\hat\pi(\eta_0 \leqslant\, -1)\,, \quad \Psi_+ := |\Lambda_+|\,\hat\pi(\eta_0 < -j). \tag{7.27}$$

Proposition 7.7. *There exists $\alpha < 1$ such that for all $\Gamma \in G$ one has the expansions (with error terms uniform in $\Gamma \in G$):*

$$\omega_-\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_-} \mathbf{1}_{\eta_x \leqslant 0}}\,;\,\eta \geqslant\, -(j-1)\Big) = \exp\big(\mathcal{K}|\Lambda_-|/L - \Phi_- - \Psi_- + O(L^\alpha)\big) \tag{7.28}$$

$$\omega_+\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_+} \mathbf{1}_{\eta_x \leqslant -1}}\,;\,\eta \geqslant\, -j\Big) = \exp\big(\Phi_+ - \Psi_+ + O(L^\alpha)\big). \tag{7.29}$$

Let us conclude the proof of Lemma 7.6 assuming for the moment the validity of Proposition 7.7. First, observe that the functions in (7.26) and (7.27) satisfy, for some $\alpha < 1$, uniformly in $\Gamma \in G$:

$$-\Phi_- + \Phi_+ = \frac{\mathcal{K}}{L}(|\Lambda_+| - |\Lambda_-|)\,\hat\pi(\eta_0 > 0)\,, \tag{7.30}$$

$$\Psi_- + \Psi_+ = |\Lambda_-|\,\hat\pi(\eta_0 \geqslant j) + |\Lambda_+|\,\hat\pi(\eta_0 \geqslant j+1). \tag{7.31}$$

From (7.28)–(7.31), setting $|\Lambda_{(j)}| = |\Lambda_-| + |\Lambda_+|$, $\delta_k(\beta) = \hat\pi(\eta_0 \geqslant k)$, and $\bar\delta_k(\beta) = L\delta_k(\beta)$:

$$\omega_-\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_-} \mathbf{1}_{\eta_x \leqslant 0}}\,;\,\eta \geqslant\, -(j-1)\Big)\;\omega_+\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_-} \mathbf{1}_{\eta_x \leqslant -1}}\,;\,\eta \geqslant\, -j\Big)$$
$$= \exp\Big(\frac{|\Lambda_{(j)}|}{L}(\mathcal{K}(1-\delta_1(\beta)) - \bar\delta_j(\beta)) + \frac{|\Lambda_+|}{L}(\mathcal{K}(-1+2\delta_1(\beta)) + \bar\delta_j(\beta) - \bar\delta_{j+1}(\beta)) + O(L^\alpha)\Big).$$

Observe that

$$\mathcal{K} = e^{-\beta(j-1-H)} \gg e^{-4\beta(j-1-H)} = Le^{-4\beta(j-1)} \asymp \bar\delta_{j-1}(\beta).$$

Therefore, for large β one sees that

$$-\mathcal{K} \leqslant \mathcal{K}(-1+2\delta_1(\beta)) + \bar\delta_j(\beta) - \bar\delta_{j+1}(\beta) \leqslant\, -\mathcal{K}/2.$$

Since the term proportional to $|\Lambda_{(j)}|$ is independent of Γ, it plays no role in (7.18). Therefore,

$$\mu(E_j\,|\,G) \leqslant \exp(O(L^\alpha)) \times \frac{\sum_{\Gamma\in E_j\cap G}\nu(\Gamma)\exp\big(-\frac{\mathcal{K}}{2L}|\Lambda_+|\big)}{\sum_{\Gamma\in G}\nu(\Gamma)\exp\big(-\frac{\mathcal{K}}{L}|\Lambda_+|\big)}, \tag{7.32}$$

where we recall that E_j is the event in (7.17), G the event in (7.21) and ν the measure in (7.24). At this point an upper bound on $\mu(E_j \mid G)$ follows from (7.32) by neglecting the negative exponent in the numerator and using Jensen's inequality for the denominator. Using also (7.25), this gives

$$\mu(E_j \mid G) \leqslant \nu(E_j) \exp\Big(\frac{\mathcal{K}}{L}\nu(|\Lambda_+|)) + O(L^\alpha)\Big). \tag{7.33}$$

It follows then from Proposition B.1 that for every β sufficiently large, for all $\varepsilon > 0$, if L is large enough:

$$\nu(E_j) \leqslant \exp(-L^{1-\varepsilon}). \tag{7.34}$$

Essentially, under ν the contour Γ behaves like a random walk and the event E_j imposes a large deviation of order $L/(\log L)^2$ which is much larger than the typical diffusive fluctuation $\sqrt{L}$. Moreover, again from Proposition B.1 one has

$$\nu(|\Lambda_+|) = O(L^{\frac{3}{2}+\varepsilon}). \tag{7.35}$$

Then (7.33),(7.34) and (7.35) end the proof of Lemma 7.6. $\square$

Proof of Proposition 7.7. Let us start with the lower bounds. Using first Jensen's inequality and then the FKG property for ω_- one has

$$\begin{aligned}
&\omega_-\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_-}\mathbf{1}_{\eta_x\,\leqslant\,0}}\,;\,\eta \geqslant\, -(j-1)\Big)\\
&\qquad\geqslant\,\exp\Big[\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_-}\omega_-(\eta_x\leqslant 0\,|\,\eta\geqslant\,-(j-1))\Big]\omega_-(\eta\geqslant\,-(j-1))\\
&\qquad\geqslant\,\exp\Big[\frac{\mathcal{K}}{L}\,|\Lambda_-| - \tilde\Phi_- - \tilde\Psi_-\Big],
\end{aligned} \tag{7.36}$$

where

$$\begin{aligned}
\tilde\Phi_- &:= \frac{\mathcal{K}}{L}\sum_{x\in\Lambda_-}\omega_-(\eta_x > 0\,|\,\eta\geqslant\,-(j-1))\,,\\
\tilde\Psi_- &:= -\sum_{x\in\Lambda_-}\log(1-\omega_-(\eta_x < -(j-1))).
\end{aligned}$$

Similarly,

$$\omega_+\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_+}\mathbf{1}_{\eta_x\,\leqslant\,-1}}\,;\,\eta\geqslant\,-j\Big)\,\geqslant\,\exp\big[\tilde\Phi_+ - \tilde\Psi_+\big], \tag{7.37}$$

where

$$\begin{aligned}
\tilde\Phi_+ &:= \frac{\mathcal{K}}{L}\sum_{x\in\Lambda_+}\omega_+(\eta_x\leqslant\,-1\,|\,\eta\geqslant\,-j)\,,\\
\tilde\Psi_+ &:= -\sum_{x\in\Lambda_+}\log(1-\omega_+(\eta_x < -j)).
\end{aligned}$$

From (7.20) one sees that both $|\Psi_- - \tilde\Psi_-|$ and $|\Psi_+ - \tilde\Psi_+|$ are $O(L^\alpha)$, for some $\alpha < 1$, uniformly in $\Gamma \in G$. Therefore the lower bound in (7.28) follows once we establish that on G

$$|\Phi_\pm - \tilde\Phi_\pm| = O(L^\alpha)\,, \tag{7.38}$$

for some $\alpha > 0$. To prove (7.38) we use the following comparison estimate. Let us consider the case $|\Phi_+ - \tilde\Phi_+|$. By FKG one has $\omega_+(\eta_x \leqslant\, -1\,|\,\eta\geqslant\,-j) \leqslant \omega_+(\eta_x\leqslant\,-1)$. On the other hand, whenever $x \in \Lambda_-$ is at distance at least L^δ, for some $\delta > 0$, from $\partial\Lambda_+$, then we claim that

$$\omega_+(\eta_x\leqslant\,-1) \leqslant \omega_+(\eta_x\leqslant\,-1\,|\,\eta\geqslant\,-j) + O(L^{\alpha-1})\,. \tag{7.39}$$

These observations and (7.20) are sufficient to prove (7.38). In turn, (7.39) is a consequence of the technique developed below, cf. the comment after (7.55).

To prove the upper bounds in (7.28) and (7.29), observe that from the FKG property of $\omega_\pm$ one has

$$\omega_-\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_-}\mathbf{1}_{\eta_x\leqslant 0}}\,;\,\eta\geqslant -(j-1)\Big)\leqslant \omega_-\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_-}\mathbf{1}_{\eta_x\leqslant 0}}\Big)\omega_-\Big(\prod_{x\in\Lambda_-}\mathbf{1}_{\eta_x\geqslant -(j-1)}\Big),$$

$$\omega_+\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_+}\mathbf{1}_{\eta_x\leqslant -1}}\,;\,\eta\geqslant -j\Big)\leqslant \omega_+\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_+}\mathbf{1}_{\eta_x\leqslant -1}}\Big)\omega_+\Big(\prod_{x\in\Lambda_+}\mathbf{1}_{\eta_x\geqslant -j}\Big).$$

Rewriting

$$\omega_-\Big(e^{\frac{\mathcal{K}}{L}\sum_{x\in\Lambda_-}\mathbf{1}_{\eta_x\leqslant 0}}\Big)=\exp\Big(\frac{\mathcal{K}}{L}|\Lambda_-|\Big)\omega_-\Big(\prod_{x\in\Lambda_-}(1-\varphi_x)\Big)$$

where $\varphi_x := 1-e^{-\frac{\mathcal{K}}{L}\mathbf{1}_{\eta_x>0}}$, and setting $\psi_x=\mathbf{1}_{\eta_x<-(j-1)}$ the bound (7.28) is then implied by

$$\omega_-\Big(\prod_{x\in\Lambda_-}(1-\varphi_x)\Big)\leqslant \exp(-\Phi_-+O(L^\alpha)), \tag{7.40}$$

$$\omega_-\Big(\prod_{x\in\Lambda_-}(1-\psi_x)\Big)\leqslant \exp(-\Psi_-+O(L^\alpha)). \tag{7.41}$$

Similarly, the bound (7.29) is implied by

$$\omega_+\Big(\prod_{x\in\Lambda_+}(1-\bar\varphi_x)\Big)\leqslant \exp(\Phi_++O(L^\alpha)), \tag{7.42}$$

$$\omega_+\Big(\prod_{x\in\Lambda_+}(1-\bar\psi_x)\Big)\leqslant \exp(-\Psi_++O(L^\alpha)), \tag{7.43}$$

with the notation $\bar\varphi_x := 1-e^{\frac{\mathcal{K}}{L}\mathbf{1}_{\eta_x<0}}$, and $\bar\psi_x=\mathbf{1}_{\eta_x<-j}$. Below, we establish (7.40)–(7.43) and (7.39). All these estimates can be achieved once one has an approximate factorization of the measure ω_+ on a mesoscopic scale L^u, $u\in(0,\frac12)$. To illustrate this point, consider the expression (7.40), and suppose the product is confined to Q_u, a square with side L^u, contained in Λ_-. Then

$$\begin{aligned}\omega_-\Big(\prod_{x\in Q_u}(1-\varphi_x)\Big)&=\sum_{A\subset Q_u}(-1)^{|A|}\,\omega_-\Big(\prod_{x\in A}\varphi_x\Big)\\&=1-\sum_{x\in Q_u}\omega_-(\varphi_x)+O\Big(\sum_{k\geqslant 2}\binom{L^{2u}}{k}L^{-k}\Big)\\&\leqslant \exp\Big(-\frac{\mathcal{K}}{L}\sum_{x\in Q_u}\omega_-(\eta_x>0)+O(L^{2(2u-1)})\Big),\end{aligned} \tag{7.44}$$

where we have separated the contributions of sets A with $|A|\leqslant 1$ and $|A|\geqslant 2$, and used the fact that $\varphi_x=\frac{\mathcal{K}}{L}\mathbf{1}_{\eta_x>0}+O(L^{-2})$. In particular if one could factorize (7.40) into a product of (7.44) over all $Q_u\subset\Lambda_-$, then the desired bound would follow using also (7.20).

To implement this idea we use the following geometric construction. Partition $\mathbb{Z}^2$ into squares P with side $r=L^u+2L^\delta$, where $0<\delta<u<\frac12$ (we assume for simplicity that L^u, L^δ are both integers). Consider squares Q of side L^u centered inside the squares P in such a way that each square Q is surrounded within P by a shell of thickness L^δ, see Figure 5. Define the set $\mathcal{S}$ of dual bonds associated to a non-zero height gradient, cf. Appendix A. The set $\mathcal{S}$ is decomposed into connected components (clusters) S. We call $\mathcal{I}(\delta)$ the collection of clusters S in $\mathcal{S}$ such that $|S| \geqslant L^\delta$. Note that a cluster may have a nonempty interior.

Consider the set of sites $V \subset \Lambda_-$ defined as what remains after we remove from Λ_- all clusters S in $\mathcal{I}(\delta)$ together with their interior. A square $Q \subset V$ is called *good* if the square $P \supset Q$ has empty intersection with $\partial\Lambda_- \cup \mathcal{I}(\delta)$; see Figure 5. We write $\mathcal{G}$ for the collection of good squares Q. The crucial observation is that if $Q \in \mathcal{G}$, then there exists a circuit $\mathcal{C}$ of

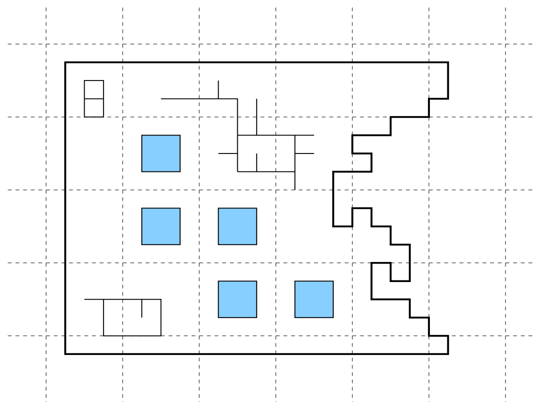

FIGURE 5. A drawing of the region Λ_-. In the background the squares P (dashed lines). The clusters inside represent the set $\mathcal{I}(\delta)$ and the shaded squares are the set $\mathcal{G}$ of good squares Q.

bonds of $\mathbb{Z}^2$ surrounding Q and contained in the square $P \supset Q$, such that $\eta_{\mathcal{C}} \equiv 0$. To see this, observe that there must be a circuit $\mathcal{C}$ of bonds surrounding Q such that gradients of η along the circuit are 0, since otherwise there would be a path of dual bonds connecting Q with P^c with cross-gradients different from zero, and therefore a cluster S with size larger than L^δ intersecting P. Now, this implies that η is constant on $\mathcal{C}$, and this constant must be zero, since otherwise Q would belong to the interior of a cluster S of size larger than L^δ because of the zero boundary condition on $\partial\Lambda_-$.

Next, we estimate $1 - \varphi_x \leqslant 1$ for all x which do not belong to some $Q \in \mathcal{G} = \mathcal{G}(\mathcal{I}(\delta))$. Therefore, summing over all possible realizations W of $\mathcal{I}(\delta)$:

$$\begin{aligned}\omega_-\Big(\prod_{x\in\Lambda_-}(1-\varphi_x)\Big) &\leqslant \sum_W \omega_-\big(\mathcal{I}(\delta)=W\big)\omega_-\Big(\prod_{Q\in\mathcal{G}}\prod_{x\in Q}(1-\varphi_x)\,|\,\mathcal{I}(\delta)=W\Big)\\ &\leqslant \sum_W \omega_-\big(\mathcal{I}(\delta)=W\big)\prod_{Q\in\mathcal{G}}\sup_{\mathcal{C}}\hat\pi^0_{\mathcal{C}}\Big(\prod_{x\in Q}(1-\varphi_x)\Big)\end{aligned} \tag{7.45}$$

where, for an arbitrary circuit $\mathcal{C}$ surrounding Q within the square $P \supset Q$ and with a slight abuse of notation, we write $\hat\pi^0_{\mathcal{C}}$ for the SOS equilibrium measure on the interior of the circuit $\mathcal{C}$ with zero boundary conditions (without floor, ceiling and no fields). With the same argument of (7.44) one has, uniformly in $\mathcal{C}$

$$\hat\pi^0_{\mathcal{C}}\Big(\prod_{x\in Q}(1-\varphi_x)\Big) \leqslant \exp\Big(-\frac{\mathcal{K}}{L}\sum_{x\in Q}\hat\pi^0_{\mathcal{C}}(\eta_x>0) + O(L^{2(2u-1)})\Big). \tag{7.46}$$

Let $\mathcal{I}(\delta) = W$ be fixed. For any fixed square $Q \in \mathcal{G}$, let $Q' \subset Q$ be the square centered inside Q in such a way that Q' is surrounded by a shell of thickness L^δ within Q. Thus, if $x \in Q'$, then

x is at distance at least L^δ from $\mathcal{C}$, and therefore as in (7.20), for any $p > 0$, uniformly in $\mathcal{C}$:

$$\hat\pi^0_{\mathcal{C}}(\eta_x > 0) = \hat\pi(\eta_x > 0) + O(L^{-p})\,, \quad x \in Q'. \tag{7.47}$$

From (7.47),

$$\hat\pi^0_{\mathcal{C}}\Big(\prod_{x\in Q}(1-\varphi_x)\Big) \leqslant \exp\Big(-\frac{\mathcal{K}}{L}\sum_{x\in Q'}\hat\pi(\eta_x>0) + O(L^{2(2u-1)}) + O(L^{-p+2u-1})\Big). \tag{7.48}$$

There are at most $O(L^{2-2u})$ squares Q. Therefore, from (7.45) one obtains

$$\omega_-\Big(\prod_{x\in\Lambda_-}(1-\varphi_x)\Big) \leqslant \sum_W \omega_-\big(\mathcal{I}(\delta)=W\big)\exp\Big(-\frac{\mathcal{K}}{L}\sum_{Q\in\mathcal{G}}\sum_{x\in Q'}\hat\pi(\eta_x>0) + O(L^{2u})\Big). \tag{7.49}$$

Next, we need to add back the contributions to the exponent in (7.49) from all removed vertices, where each vertex contributes at most $1/L$. The contribution of a single removed shell $P\setminus Q'$ is $O(L^{\delta+u-1})$, and they are at most $O(L^{2-2u})$, so that all removed shells give at most $O(L^{1+\delta-u}) = O(L^\alpha)$ for $\alpha < 1$ since $u > \delta$. To estimate the contribution from all other removed sites, we observe that a site can be removed if it belongs to a square P that intersects either the boundary $\partial\Lambda_-$ or the clusters of $\mathcal{I}(\delta)$, or if it belongs to the interior of a cluster of $\mathcal{I}(\delta)$. If $A(\mathcal{I}(\delta))$ denotes the total number of sites in the interior of the clusters $S \in \mathcal{I}(\delta)$, then these contribute at most $\mathcal{K}L^{-1}\times A(\mathcal{I}(\delta))$. Moreover, one has at most $L^{2u}\times|\mathcal{I}(\delta)|$ sites that can be removed from intersections with $\mathcal{I}(\delta)$. These contribute at most $\mathcal{K}L^{2u-1}|\mathcal{I}(\delta)|$. Finally, one estimates roughly by $L^{2u}|\partial\Lambda_-| = O(L^{2u+1+\varepsilon_0})$ the number of sites removed from squares intersecting $\partial\Lambda_-$, since on the event G one has $|\partial\Lambda_-| = O(L^{1+\varepsilon_0})$. Thus, the contribution from the boundary squares is $O(L^{2u+\varepsilon_0}) = O(L^\alpha)$, if $2u+\varepsilon_0 < 1$. Therefore, using $\mathcal{K}\leqslant 1$:

$$\frac{\mathcal{K}}{L}\sum_{Q\in\mathcal{G}}\sum_{x\in Q'}\hat\pi(\eta_x>0) \geqslant \Phi_- - L^{-1}A(\mathcal{I}(\delta)) - L^{2u-1}|\mathcal{I}(\delta)| + O(L^\alpha). \tag{7.50}$$

Thus, we have obtained

$$\omega_-\Big(\prod_{x\in\Lambda_-}(1-\varphi_x)\Big) \leqslant \exp\big(-\Phi_- + O(L^\alpha)\big)\,\omega_-\left[\exp\big(L^{-1}A(\mathcal{I}(\delta)) + L^{2u-1}|\mathcal{I}(\delta)|\big)\right]. \tag{7.51}$$

If $\{S_i\}_{i=1}^m$ denotes the collection of clusters of $\mathcal{I}(\delta)$, with $|\mathcal{I}(\delta)| = \sum_i |S_i|$, then

$$A(\mathcal{I}(\delta)) \leqslant \frac14\sum_i |S_i|^2 \leqslant \frac14\Big(\sum_i|S_i|\Big)^2, \tag{7.52}$$

where the last bound follows from $\sum_i x_i^2 \leqslant (\sum_i x_i)^2$ for all $x_i \geqslant 0$. Recalling that $|S_i|\geqslant L^\delta$ for all i, using (A.2), letting m represent the number of clusters $S_1,\dots,S_m$, and summing over their starting points $x_1,\dots,x_m$, one has the estimate

$$\omega_-\Big(\sum_i|S_i|\geqslant k\Big) \leqslant \sum_{m\,\geqslant\,1}\sum_{x_1,\dots,x_m}\sum_{S_1\ni x_1}\cdots\sum_{S_m\ni x_m} C\,e^{-\beta(|S_1|+\cdots|S_m|)/2}\chi(S_1,\dots,S_m)$$

where $\chi(S_1,\dots,S_m) = 1$ if $|S_i| \geqslant L^\delta$ for all $i = 1,\dots,m$ and $|S_1| + \cdots |S_m| \geqslant k$, and $\chi(S_1,\dots,S_m) = 0$ otherwise. Therefore,

$$\begin{aligned}\omega_-\Big(\sum_i |S_i| \geqslant k\Big) &\leqslant C\,e^{-\beta k/4} \sum_{m\geqslant 1}\Bigg(\sum_x \sum_{S\ni x,\,|S|\geqslant L^\delta} e^{-\beta|S|/4}\Bigg)^m \\ &\leqslant C\,e^{-\beta k/4}\sum_{m\geqslant 1} L^{2m}\Bigg(\sum_{j\geqslant L^\delta} C^j e^{-\beta j/4}\Bigg)^m \leqslant e^{-\beta k/4}\,, \end{aligned} \tag{7.53}$$

for any β large enough and for all L sufficiently large. From (7.52) and (7.53) one has

$$\omega_-(A(\mathcal{I}(\delta)) \geqslant \ell) \leqslant \omega_-\Big(\sum_i |S_i| \geqslant 2\sqrt{\ell}\Big) \leqslant e^{-\beta\sqrt{\ell}/2}$$

for all $\ell > 0$ and therefore

$$\omega_-\left[\exp\left(2L^{-1}A(\mathcal{I}(\delta))\right]\right. \leqslant \sum_{\ell=0}^{L^2}\exp\left(2\ell/L - \beta\sqrt{\ell}/4\right) \leqslant L^2 + 1, \tag{7.54}$$

since $2\ell/L \leqslant \beta\sqrt{\ell}/4$, for β large and $\ell \leqslant L^2$. Using (7.54), a Cauchy-Schwarz inequality and (7.53), it follows that

$$\omega_-\left[\exp\left(L^{-1}A(\mathcal{I}(\delta)) + L^{2u-1}|\mathcal{I}(\delta)|\right)\right] \leqslant CL \leqslant e^{O(L^\alpha)} \tag{7.55}$$

for any $\alpha < 1$. This ends the proof of (7.40). To prove (7.42) one repeats the same argument with the region Λ_- replaced by Λ_+.

We turn to the proof of the estimates (7.41) and (7.43). A minor modification of the same argument proves also the inequality (7.39). Here one has to replace the expansion (7.46) by the following bound:

$$\hat\pi^0_{\mathcal{C}}\Big(\prod_{x\in Q}(1-\psi_x)\Big) \leqslant \exp\Big(-\sum_{x\in Q}\hat\pi^0_{\mathcal{C}}(\psi_x) + O(L^{-\frac{3}{2}+2u+c(\beta)}) + O(L^{6u-3})\Big), \tag{7.56}$$

where $\psi_x = \mathbf{1}_{\eta_x \geqslant j}$ and $c(\beta) > 0$ can be made arbitrarily small by taking β large enough. Once this estimate (together with the corresponding statement for $\bar\psi_x = \mathbf{1}_{\eta_x > j}$) is available, it is not hard to check that exactly the same arguments we used to prove (7.40) and (7.42) allow one to conclude. Here the term $\mathcal{K}L^{-1}\hat\pi(\eta_x > 0)$ appearing in (7.50) must be replaced by $\hat\pi(\eta_x \geqslant j)$, which (thanks to $j \geqslant H+1$) is again less than L^{-1} for β large enough by (7.19). In particular, one can use the argument in (7.54)–(7.55) to conclude as above.

It remains to prove (7.56). We cannot proceed as in (7.44) since ψ_x is not pointwise $O(1/L)$. From Bonferroni's inequality (inclusion-exclusion principle), one has

$$\begin{aligned}\hat\pi^0_{\mathcal{C}}\Big(\prod_{x\in Q}(1-\psi_x)\Big) &= \sum_{A\subset Q}(-1)^{|A|}\,\hat\pi^0_{\mathcal{C}}\Big(\prod_{x\in A}\psi_x\Big) \\ &\leqslant 1 - \sum_{x\in Q}\hat\pi^0_{\mathcal{C}}(\psi_x) + \frac{1}{2}\sum_{\substack{x,y\in Q:\\ x\neq y}}\hat\pi^0_{\mathcal{C}}(\psi_x\psi_y). \end{aligned} \tag{7.57}$$

Next, observe that

$$\sum_{\substack{x,y\in Q:\\ x\neq y}}\hat\pi^0_{\mathcal{C}}(\psi_x\psi_y) = \Big(\sum_{x\in Q}\hat\pi^0_{\mathcal{C}}(\psi_x)\Big)^2 + \sum_{\substack{x,y\in Q:\\ x\neq y}}\hat\pi^0_{\mathcal{C}}(\psi_x;\psi_y) + O(L^{-2+2u}) \tag{7.58}$$

where $\hat\pi^0_{\mathcal{C}}(\psi_x;\psi_y) := \hat\pi^0_{\mathcal{C}}(\psi_x\psi_y) - \hat\pi^0_{\mathcal{C}}(\psi_x)\hat\pi^0_{\mathcal{C}}(\psi_y)$, and we use $\hat\pi^0_{\mathcal{C}}(\psi_x) = O(1/L)$. We need the following bound. For some $c(\beta) \to 0$ as $\beta \to \infty$, one has

$$\sum_{\substack{x,y\in Q:\\ x\neq y}} \hat\pi^0_{\mathcal{C}}(\psi_x;\psi_y) = O(L^{-3/2+2u+c(\beta)}). \tag{7.59}$$

The bound (7.56) follows immediately from (7.57)–(7.59) and the fact that $\left(\sum_{x\in Q} \hat\pi^0_{\mathcal{C}}(\psi_x)\right)^3 = O(L^{6u-3})$.

To prove (7.59), first notice that by exponential decay of correlations [11]:

$$\hat\pi^0_{\mathcal{C}}(\psi_x;\psi_y) \leqslant c_1 e^{-c_2|x-y|}\,,$$

for some constants $c_1, c_2 > 0$. Therefore (7.59) follows if we prove that for any constant $C > 0$,

$$\sum_{0\neq|y|\,\leqslant\, C\log L} \hat\pi^0_{\mathcal{C}}(\psi_0;\psi_y) = O(L^{-3/2+c(\beta)}).$$

In particular, it suffices to show that uniformly in $y \neq 0$:

$$\hat\pi^0_{\mathcal{C}}(\psi_0\psi_y) = O(L^{-3/2+c'(\beta)}) \tag{7.60}$$

for some constant $c'(\beta) \to 0$ as $\beta \to \infty$. The proof of (7.60) goes as follows. Let E_k denote the event that there exists some k-contour γ that contains both $0, y$. Then $E_{k+1} \subset E_k$, $k \geqslant 1$, and

$$\hat\pi^0_{\mathcal{C}}(\psi_0\psi_y) = \hat\pi^0_{\mathcal{C}}(\psi_0\psi_y; E_1^c) + \sum_{k=1}^{j} \hat\pi^0_{\mathcal{C}}(\psi_0\psi_y; E_k \cap E_{k+1}^c) + \hat\pi^0_{\mathcal{C}}(\psi_0\psi_y; E_{j+1}).$$

Now, if $\psi_0\psi_y \cap E_1^c$ occurs, then there must be two separate families of nested contours reaching level j, one around 0 and the other around y. By repeating the argument in the proof of Proposition 3.9, one has

$$\hat\pi^0_{\mathcal{C}}(\psi_0\psi_y; E_1^c) = O(e^{-8\beta j}) = O(L^{-2}).$$

If $\psi_0\psi_y \cap E_k \cap E_{k+1}^c$ occurs then there must be nested contours around 0 and around y separately from level $k+1$ to level j and there must be nested contours from level 1 to level k comprising both 0 and y. In this case the argument in the proof of Proposition 3.9 yields

$$\hat\pi^0_{\mathcal{C}}(\psi_0\psi_y; E_k \cap E_{k+1}^c) = O(e^{-8\beta(j-k)} e^{-\beta(1-c(\beta))k\ell_y})$$

where ℓ_y denotes the length of the shortest contour comprising both $0, y$ and $c(\beta)$ decays as fast as $1/\beta$. Since $\ell_y \geqslant 6$, the above expression is $O(e^{-6\beta(1-c(\beta))j}) = O(L^{-3/2+c'(\beta)})$ for every $k \leqslant j$. Finally, the same argument shows that

$$\hat\pi^0_{\mathcal{C}}(\psi_0\psi_y; E_{j+1}) = O(e^{-6\beta(1-c(\beta))j}) = O(L^{-3/2+c'(\beta)}).$$

These estimates imply (7.60). This ends the proof of Proposition 7.7. ☐

8. Mixing time in absence of entropic repulsion: Proof of Theorem 3

Like for Theorem 1, it is sufficient to give the proof when $n^+ = \log L$, the general case following easily (one needs to generalize the approach of Section 6.3 in the obvious way). For simplicity of notations, we call the SOS equilibrium measure with zero b.c. on $\partial\Lambda_L$ and floor/ceiling at $\pm\log L$ simply π. Recall the definition (5.1) of the diagonal lines R_i and define, for $1 \leqslant j \leqslant N = (2L-1)/L^{1/2+\varepsilon}$ (assume for simplicity that N and $L^{1/2+\varepsilon}$ are integers), the subset W_j of Λ_L as

$$W_j = \bigcup_{i\in\mathcal{I}_j} R_i, \quad \mathcal{I}_j = \left\{\frac{j-1}{2}L^{1/2+\varepsilon} < i \leqslant \frac{j+1}{2}L^{1/2+\varepsilon}\right\}.$$

Note that $W_j \cap W_{j+1}$ is a roughly rectangular-shaped region of smaller side of order $L^{1/2+\varepsilon}$. Let also S_j denote the "brick" with horizontal projection W_j and floor/ceiling at $\pm \log L$. From (2.8) and symmetry, we know that we have only to show that

$$\|\mu_T^{\sqcap} - \pi\| \leqslant L^{-3} \tag{8.1}$$

with $T = \exp(c\beta L^{1/2+2\varepsilon})$ and some large constant c.

The proof is somewhat similar (but definitely simpler) to that of Lemma 6.6, so we will be very sketchy. The simplification is that, since the floor at $-\log L$ has essentially no effect at equilibrium, it is not necessary to introduce the field term (6.1) to compensate the entropic repulsion.

We apply Theorem 2.2 with the following censoring protocol. We let $\Delta T = T/N$ and we let evolve first the brick S_1 for a time-lag ΔT, then S_2 for another time-lag ΔT, and so on up to S_N. From Proposition 2.3 (which is immediately adapted to the case where Λ is not *exactly* a $L \times m$ rectangle but rather is *included* in some, possibly tilted, $L \times m$ rectangle) we have that the mixing time in each brick S_j, uniformly on the b.c. around it, is $\exp(O(\beta n^+ L^{1/2+\varepsilon}))$. Therefore, if c in the definition of T is sufficiently large, we can assume (modulo a negligible error term) that after the j^{th} time-lag the j^{th} brick is exactly at equilibrium, with 0 b.c. on $\partial W_j \cap \partial \Lambda_L$, b.c. $n^+ = \log L$ on $\partial W_j \cap W_{j+1}$ and, on $\partial W_j \cap W_{j-1}$, a b.c. determined by the result of the evolution in the $(j-1)^{th}$ time-lag. Theorem 2.2 then guarantees that the l.h.s. of (8.1) is smaller than $\|\tilde{\mu}_T^{\sqcap} - \pi\|$, with $\tilde{\mu}_T^{\sqcap}$ the law at time T of the censored dynamics. The inequality (8.1) then follows (via a repeated application of DLR) provided that one proves that, if π_j denotes the equilibrium on $U_j = W_1 \cup \cdots \cup W_j$ with 0 b.c. on $\partial U_j \cap \partial \Lambda_L$ and n^+ b.c. on $\partial U_j \cap W_{j+1}$, then

$$\|\pi_j - \pi\|_{U_{j-1}} = O(L^{-4}) \tag{8.2}$$

i.e. the marginals of the two measures on U_{j-1} are very close.

In analogy with the way Theorem 6.12 follows from Lemma 7.1 (cf. Section 7.1), to get (8.2) it is sufficient to prove that the open 1-contour does not intersect W_{j-1}, except with probability $O(L^{-C})$. In turn (and in analogy to how Lemma 7.1 follows from Lemma 7.6), the desired upper bound on the deviation of the 1-contour follows if we prove the following. Consider a diagonal line R_i, with $i \geqslant L^{1/2+\varepsilon}$. Let $\Lambda' = \cup_{a \leqslant i} R_a$ and let ρ be a chain of sites in Λ', connecting two adjacent sides of Λ_L, and at distance at most L^ε from R_i, see Figure 6.

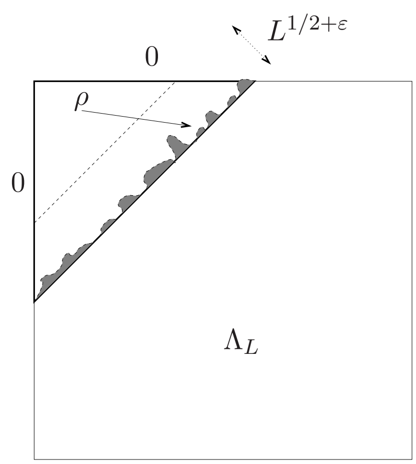

FIGURE 6. A drawing of the triangular region Λ' and of the chain ρ.

The chain ρ disconnects Λ_L into two subsets and call Λ_- the one containing the North-West corner of Λ_L. Let π' be the SOS measure on Λ_-, with 0 b.c. on $\partial\Lambda_- \cap \partial\Lambda_L$ and 1 b.c. on ρ. Then, the π'-probability that the unique 1-contour reaches distance $L^{1/2+\varepsilon}$ from ρ is smaller than any inverse power of L.

This is much easier to prove than the somewhat similar estimate of Lemma 7.6. The reason is that, since the fields (6.1) are absent and the floor has a negligible effect (recall that the floor was instead at height zero in Lemma 7.6), the desired estimate follows directly from a suitable modification of Proposition B.1, where the square Q_L is replaced by a triangular domain. ☐

Appendix A. Peierls' estimates and low-temperature expansion

Here we collect some rather standard facts concerning the low temperature expansion of the SOS model. With a small abuse of notation let Z_Λ be the partition function corresponding to the measure $\hat{\pi}^0_\Lambda$. Following [11, Section 2] we will write Z_Λ as a sum over compatible cluster configurations.

Definition A.1. *A* cluster X *is a tuple* $(\gamma, h_1, \dots, h_{|\gamma|})$*, where* γ *is a finite connected set of dual lattice bonds, and* $h_i \in \mathbb{Z}\setminus\{0\}$*. A cluster configuration is a collection of clusters* $\{X_1, \dots, X_m\}$*.*

Let Ω^0_Λ be the set of height functions $\eta \in \mathbb{Z}^{\mathbb{Z}^2}$ with $\eta_x = 0$ for every $x \notin \Lambda$. Given $\eta \in \Omega^0_\Lambda$ one can define the associated cluster configuration $\{X_1, \dots, X_m\}$ as follows. Fix an arbitrary orientation of the edges $e = (x, y)$ of $\mathbb{Z}^2$. Let $\mathcal{S} = \mathcal{S}(\eta)$ be the collection of all dual edges e' such that the gradient of η along the edge $e = (x, y)$ crossing e' satisfies $h_e := \eta_y - \eta_x \neq 0$. Let $\gamma_1, \dots, \gamma_m$ denote the connected components of $\mathcal{S}$. For each $j = 1, \dots, m$ let $X_j = (\gamma_j, \{h_e\})$ denote the associated cluster, where $\{h_e\}$ denotes the collection of gradients of η along edges e that cross a dual edge $e' \in \gamma_j$. This defines an injective map of Ω^0_Λ into the set of cluster configurations. The range of this map will be denoted $\mathcal{D}(\Lambda)$. Thus,

$$Z_\Lambda = \sum_{\{X_1,\dots,X_m\}\in\mathcal{D}(\Lambda)} \prod_{j=1}^m \rho(X_j)\,, \qquad \rho(X_j) = \exp\Big[-\beta\sum_e |h_e|\Big] \tag{A.1}$$

where the sum over e extends over all $|\gamma_j|$ edges e which cross a dual edge $e' \in \gamma_j$.

A.1. **A Peierls' estimate.** As above, $\mathcal{S}$ denotes the random set of dual edges crossing a nonzero gradient.

Lemma A.2. *There exists* $\beta_0 > 0$ *such that for all* $\beta \geqslant \beta_0$*, for all finite connected* $\Lambda \subset \mathbb{Z}^2$*, and all set* V *of dual edges,*

$$\hat{\pi}^0_\Lambda(\mathcal{S} \supset V) \leqslant e^{-(\beta-\beta_0)|V|}. \tag{A.2}$$

Proof. We suppose that V is connected, since the general case follows by a standard generalization. Let e' be a dual edge in V and let $\mathcal{S}_0$ denote the largest connected component of $\mathcal{S}$ containing e'. Then

$$\hat{\pi}^0_\Lambda(V \subset \mathcal{S}) \leqslant \sum_{S\colon S\supset V} \hat{\pi}^0_\Lambda(\mathcal{S}_0 = S)$$

where the sum is over all connected sets S of dual edges, such that $S \supset V$. Any $\eta \in \Omega^0_\Lambda$ such that $\mathcal{S}_0 = S$ corresponds to a cluster configuration $\{X_S, X_1, \dots, X_m\} \in \mathcal{D}(\Lambda)$, where X_S is a cluster of the form $X_S = (S, h_1, \dots, h_{|S|})$. For a fixed S one has $\sum_{h_1\neq 0,\dots,h_{|S|}\neq 0} \rho(X_S) \leqslant (4e^{-\beta})^{|S|}$, if $\beta \geqslant \log 2$. Therefore, using (A.1), neglecting the constraints on X_S, one has

$$\hat{\pi}^0_\Lambda(\mathcal{S}_0 = S) \leqslant (4e^{-\beta})^{|S|}. \tag{A.3}$$

Summing over all S as above and estimating by C^ℓ the number of connected $S \ni e'$ with $|S| = \ell$ gives

$$\hat{\pi}^0_\Lambda (V \subset \mathcal{S}) \leqslant \sum_{\ell \geqslant |V|} (4Ce^{-\beta})^\ell \leqslant e^{-(\beta-\beta_0)|V|}.$$

□

A.2. **Cluster expansion.** We shall use a standard expansion for partition functions, adapted from [23,37]. For $U \subset \Lambda$, we write $Z_{\Lambda,U}$ for the partition function with the sum over η restricted to those $\eta \in \Omega^0_\Lambda$ such that $\eta_x \geqslant 0$ for all $x \in U$. One can write again $Z_{\Lambda,U}$ as in (A.1), provided the set of configurations $\mathcal{D}$ is defined as the set $\mathcal{D} = \mathcal{D}(\Lambda, U)$ that is in one-to-one correspondence with the $\eta \in \Omega^0_\Lambda$ such that $\eta_x \geqslant 0$ for all $x \in U$.

Lemma A.3. *There exists β_0 such that for all $\beta \geqslant \beta_0$, for all finite connected $\Lambda \subset \mathbb{Z}^2$ and $U \subset \Lambda$,:*

$$\log Z_{\Lambda,U} = \sum_{V\subset\Lambda} \varphi_U(V), \tag{A.4}$$

where the potentials $\varphi_U(V)$ satisfy

(i) $\varphi_U(V) = 0$ if V is not connected.

(ii) $\varphi_U(V) = \varphi_0(V)$ if $\mathrm{dist}(V, U) \neq 0$, for some shift invariant potential $V \mapsto \varphi_0(V)$ that is

$$\varphi_0(V) = \varphi_0(V + x) \quad \forall\, x \in \mathbb{Z}^2 \ .$$

(iii) There exists a constant $\beta_0 > 0$ such that

$$\sup_{\Lambda\supset V} \sup_{U} |\varphi_U(V)| \leqslant \exp(-(\beta - \beta_0)\, d(V))$$

where $d(V)$ is the cardinality of the smallest connected set of bonds of $\mathbb{Z}^2$ containing all the boundary bonds of V (i.e., bonds connecting V to V^c).

Proof. We shall apply the main theorem from [37]. Let $\mathcal{L}(\Lambda, U)$ denote the set of clusters X such that the cluster configuration $\{X\}$ belongs to $\mathcal{D}(\Lambda, U)$. If $X, X' \in \mathcal{L}(\Lambda, U)$, then X, X' are called *compatible*, in symbols $X \sim X'$, iff $\gamma \cup \gamma'$ is not a connected set of dual edges, where γ, γ' denote the geometric part of X, X' respectively. Otherwise, X, X' are said to be incompatible, in symbols $X \nsim X'$. Following [37], we define $\mathcal{C}$ as the set of all cluster configurations C that cannot be decomposed as $C = C_1 \cup C_2$ with two nonempty cluster configurations C_1, C_2 such that $\{X_1, X_2\}$ is compatible for every $X_1 \in C_1$ and $X_2 \in C_2$. For a cluster $X = (\gamma, h_1, \dots, h_{|\gamma|})$, define the function $a(X) = \lambda|X|$, where $\lambda > 0$ is to be specified later and $|X| := \sum_{i=1}^{|\gamma|} |h_i|$. Note that, for a fixed $X = (\gamma, h_1, \dots, h_{|\gamma|})$, one has

$$\sum_{X'\colon X'\nsim X} e^{2\lambda|X'|}\rho(X') \leqslant \sum_{\gamma'\colon \gamma\cup\gamma' \text{connected}} c(\beta, \lambda)e^{-(\beta-2\lambda)|\gamma'|} \leqslant c'(\beta, \lambda)|\gamma|, \tag{A.5}$$

where e.g. $c(\beta, \lambda) = 2(1 - e^{-(\beta-2\lambda)})^{-1}$ and $c'(\beta, \lambda) = 3e^{-(\beta-2\lambda)}c(\beta, \lambda)$. So if $\beta \geqslant 2\lambda + 1$, and λ is larger than some absolute value λ_0, (A.5) implies

$$\sum_{X'\colon X'\nsim X} e^{2\lambda|X'|}\rho(X') \leqslant a(X)\,. \tag{A.6}$$

Equation (A.6) corresponds to Eq. (1) in [37]. The main theorem there then allows one to write

$$\log Z_{\Lambda,U} = \sum_{C\colon C\subset\mathcal{L}(\Lambda,U)} \Phi(C), \tag{A.7}$$

for a function Φ on cluster configurations satisfying $\Phi(C) = 0$ if $C \notin \mathcal{C}$ and

$$\sum_{C:\, C \not\sim X} |\Phi(C)| e^{a(C)} \leqslant a(X), \tag{A.8}$$

for every cluster X, where $a(C) := \sum_{i=1}^{n} a(X_i)$ if $C = \{X_1, \dots, X_n\}$ and the notation $C \not\sim X$ indicates that $X_i \not\sim X$ for some $X_i \in C$. The potentials Φ depend on U but for lightness of notations we keep this implicit. Taking X to be the elementary unit square cluster such that $|X| = 4$ in (A.8) one finds in particular that for every cluster configuration C one has

$$|\Phi(C)| \leqslant 4e^{-a(C)}. \tag{A.9}$$

To write $Z_{\Lambda,U}$ as in (A.4) we follow [23, Section 3.9]. For any cluster configuration $C \in \mathcal{C}$, $C = \{X_1, \dots, X_n\}$ with $X_i = (\gamma_i, h_1, \dots, h_{|\gamma_i|})$, write C_g for the geometric part of C, i.e. $C_g = (\gamma_1, \dots, \gamma_n)$. For any $G := (\gamma_1, \dots, \gamma_n)$ define

$$\psi(G) = \sum_{C \in \mathcal{C}:\, C_g = G} \Phi(C).$$

Using (A.9), if $\lambda \geqslant \lambda_0$, one has

$$|\psi(G)| \leqslant 4e^{-\frac{\lambda}{2}\sum_i |\gamma_i|}. \tag{A.10}$$

Finally, set

$$\varphi_U(V) = \sum_{\substack{G=(\gamma_1,\dots,\gamma_n):\\ \cup_i \mathrm{Int}\gamma_i = V}} \psi(G). \tag{A.11}$$

From (A.7) one obtains the expansion (A.4). The properties (i)-(ii)-(iii) follow as in [23] from an explicit representation of the function $\Phi(C)$, and from the exponential decay (A.10). ☐

A.3. **Distribution of an open contour.** Here we apply the expansion of Lemma A.3 to derive an expression for the law of an open contour in the presence of a stepped boundary condition. Suppose a finite connected $\Lambda \subset \mathbb{Z}^2$ is given together with a boundary condition ξ with values in $\{0, 1\}$ and such that it induces a *unique* open 1-contour γ. If $\gamma = \Gamma$, for some connected set of dual edges Γ, then Λ is partitioned into two connected regions Λ_+, Λ_- separated by Γ. Moreover,

$$\hat{\pi}^{\xi}_{\Lambda}(\gamma = \Gamma) \propto e^{-\beta|\Gamma|} Z_{\Lambda_-, \Delta^-_\Gamma} Z_{\Lambda_+, \Delta^+_\Gamma}, \tag{A.12}$$

where $\Delta^{\pm}_\Gamma$ are the sets defined after (7.17), and we use the notation $Z_{\Lambda,U}$ introduced in Lemma A.3. By expanding the partition functions as in (A.4), and retaining only terms depending on Γ, one finds that

$$\hat{\pi}^{\xi}_{\Lambda}(\gamma = \Gamma) \propto \exp(-\beta|\Gamma| + \Psi_\Lambda(\Gamma)), \tag{A.13}$$

where

$$\Psi_\Lambda(\Gamma) = -\sum_{\substack{V \subset \Lambda\\ V \cap \Gamma \neq \emptyset}} \varphi_0(V) + \sum_{\substack{V \subset \Lambda_+\\ V \cap \Gamma \neq \emptyset}} \varphi_{\Delta^+_\Gamma}(V) + \sum_{\substack{V \subset \Lambda_-\\ V \cap \Gamma \neq \emptyset}} \varphi_{\Delta^-_\Gamma}(V).$$

Here, the notation $V \cap \Gamma \neq \emptyset$ simply means that $V \cap (\Delta^-_\Gamma \cup \Delta^+_\Gamma) \neq \emptyset$. It is convenient to rewrite this expansion in the form

$$\Psi_\Lambda(\Gamma) = \sum_{\substack{V \subset \Lambda\\ V \cap \Gamma \neq \emptyset}} \phi(V; \Gamma) \tag{A.14}$$

where the "decorations" $\{\phi(V;\Gamma)\}_{V \subset \Lambda}$ satisfy (cf. Lemma A.3):

(i) $\phi(V; \Gamma) = 0$ if V is not connected.

(ii) ϕ is shift invariant in the sense that

$$\phi(V;\Gamma) = \phi(V+x;\Gamma+x) \quad \forall\, x \in \mathbb{Z}^2.$$

(iii) There exists a constant $\beta_0 > 0$ such that

$$\sup_{\Gamma} |\phi(V;\Gamma)| \leqslant \exp(-(\beta-\beta_0)\, d(V))$$

where $d(V)$ is defined as in Lemma A.3.

It is standard to check that these properties imply the existence of β_0 such that, for any $\beta \geqslant \beta_0$ and any $\ell \geqslant 1$,

$$\sum_{\substack{V \ni 0 \\ d(V) \geqslant \ell}} \sup_{\Gamma} |\phi(V;\Gamma)| \leqslant \exp(-(\beta-\beta_0)\ell). \tag{A.15}$$

Appendix B. Large deviations of the contour

We begin by fixing some notation. $\mathbb{S} = \{1, 2 \dots, L\} \times \mathbb{Z}$ will denote the infinite vertical strip of width L. We denote by A, B the points of coordinates $(1, 0)$ and $(L, 0)$ respectively. The $L \times L$ square with corners A, B, C, D, where $C = (L, L)$ and $D = (1, L)$ will be denoted by Q_L. Next we fix an open contour Γ_* inside Q_L joining A with B with the property that Γ_* stays above the line at zero height and does not reach height L^δ, for some $\delta < 1/2$ that in the applications will be taken small. The region inside $\mathbb{S}$ above Γ_* is denoted by Λ and we set $Q = Q_L \cap \Lambda$. We let ν_Q be the law of the open 1-contour Γ joining A with B, for the SOS model without floor/ceiling in Q, with 1 b.c. along Γ_* and 0 b.c. otherwise. We know that ν_Q can be written as

$$\nu_Q(\Gamma) \propto \exp(-\beta|\Gamma| + \Psi_Q(\Gamma)) \tag{B.1}$$

where Ψ_Q is the function appearing in (A.14). Fix $a \in (1/2, 1)$ and $\ell \in [L^a, L/\log(L)^2]$ and define E_ℓ as the event that the path Γ reaches height ℓ (note that $\ell \gg L^\delta$).

Proposition B.1. *Uniformly in Γ_* as above, there exists β_0 independent of $(\ell,\, L)$ such that, for all $\beta > \beta_0$ and all L large enough*

$$\nu_Q(E_\ell) \leqslant c' \exp(-c\,\ell^2/L)$$

for some constants c, c'.

B.1. **Proof of Proposition B.1.** As a first preliminary step we remove the dependence on the upper boundary of Q. Let ν_Λ be the probability distribution on contours in Λ joining A, B given by

$$\nu_\Lambda(\Gamma) \propto \exp\Big(-\beta|\Gamma| + \Psi_\Lambda(\Gamma)\Big).$$

Claim B.2. *For any β large enough*

$$\nu_Q(E_\ell) \leqslant 3\, \nu_\Lambda(E_\ell) + e^{-cL}$$

for a suitable constant $c = c(\beta)$.

Proof of the Claim. Let $\mathcal{G}_0$ and $\mathcal{G}_1$ be the set of contours which stay below height $L - \log(L)^2$ and height L respectively. Then

$$\nu_Q(E_\ell) \leqslant \frac{\nu_\Lambda(\chi_{E_\ell}\, \chi_{\mathcal{G}_0}\, e^{\Delta\Psi(\Gamma)})}{\nu_\Lambda(\chi_{\mathcal{G}_0}\, e^{\Delta\Psi(\Gamma)})} + \frac{\nu_\Lambda(\chi_{E_\ell}\, (1-\chi_{\mathcal{G}_0})\, \chi_{\mathcal{G}_1}\, e^{\Delta\Psi(\Gamma)})}{\nu_\Lambda(\chi_{\mathcal{G}_0}\, e^{\Delta\Psi(\Gamma)})} \tag{B.2}$$

where

$$\Delta\Psi(\Gamma) = \Psi_Q(\Gamma) - \Psi_\Lambda(\Gamma)$$

and the inequality sign comes from restricting the average in the denominator from contours in $\mathcal{G}_1$ to contours $\mathcal{G}_0$. Since $\min_{\Gamma\in\mathcal{G}_0^c}|\Gamma| - \min_{\Gamma\in\mathcal{G}_0}|\Gamma| \geqslant L$, a standard Peierls argument shows that

$$\nu_\Lambda(\mathcal{G}_0^c) \leqslant e^{-(\beta-\beta_0)L}$$

for some β_0. Moreover, thanks to the exponential decay of the decorations (A.14),

$$|\Delta\Psi(\Gamma)| \leqslant \tfrac{1}{2} \quad \forall\,\Gamma\in\mathcal{G}_0 \quad \text{and} \quad |\Delta\Psi(\Gamma)| \leqslant e^{-(\beta-\beta_0)}L \quad \forall\,\Gamma\in\mathcal{G}_1.$$

Therefore the first term in the r.h.s. of (B.2) is smaller than $3\,\nu_\Lambda(E_\ell)$ while the second one is bounded from above by e^{-cL} for some constant $c = c(\beta)$ diverging as $\beta\to\infty$. □

Back to the proof of the proposition: since the event E_ℓ is increasing, we can change the b.c. from 0 to 1 along the lateral sides of Λ, up to height $(3/4)\ell$ (note that in this situation the endpoints of Γ are shifted upward by $(3/4)\ell$). We still call ν_Λ the measure of Γ in this situation. Again by FKG, we have

$$\nu_\Lambda(E_\ell) \leqslant \frac{\nu_\Lambda(E_\ell; G^+)}{\nu_\Lambda(G^+)} \tag{B.3}$$

where G^+ is the increasing event that Γ stays at distance at least L^ε from Γ_*, for some small but positive constant ε.

Thanks to the decay properties of the potentials $\phi(V;\Gamma)$, for every Γ in G^+ we can replace $\Psi_Q(\Gamma)$ with $\Psi_{\mathbb{S}}(\Gamma)$, up to a negligible error term. Then, the ratio (B.3) equals

$$(1+o(1))\frac{\nu_{\mathbb{S}}(E_\ell; G^+)}{\nu_{\mathbb{S}}(G^+)} \leqslant (1+o(1))\frac{\nu_{\mathbb{S}}(E_\ell)}{\nu_{\mathbb{S}}(G^+)} \tag{B.4}$$

with $\nu_{\mathbb{S}}$ the measure of the contour for SOS in the strip $\mathbb{S}$, with 0 b.c. above $A+(0,(3/4)\ell), B+(0,(3/4)\ell)$ and 1 b.c. below it.

Note that, if the complementary event $(G^+)^c$ happens, it means that the contour Γ makes a downward deviation at least $(1/2)\ell$ from its natural height $(3/4)\ell$. Therefore, $\nu_{\mathbb{S}}(G^+) \geqslant 1 - \nu_{\mathbb{S}}(E_\ell)$. As a consequence, it suffices to prove:

Claim B.3. *For any $c > 0$ and all β large enough depending on c, one has $\nu_{\mathbb{S}}(E_\ell) \leqslant e^{-c\ell^2/L}$.*

Proof of the Claim. By translation invariance, we can assume that the path Γ starts at $A = (1,0)$, ends at $B = (L,0)$ and and replace E_ℓ with $E_{\ell/4}$. We would like to appeal to the results of Section 4.15 in [23]. For this purpose we need to tackle the fact that decorations touching the boundary of $\mathbb{S}$ may behave differently from decorations inside $\mathbb{S}$. We therefore introduce a third (!) probability measure on *all* paths Γ between A and B (even those going outside $\mathbb{S}$) denoted simply by $\mathbb{P}(\cdot)$ and corresponding to the weight $e^{-\beta|\Gamma|+\Psi_{\mathbb{Z}^2}(\Gamma)}$ and we write

$$\begin{aligned}&\nu_{\mathbb{S}}(\Gamma \text{ reaches height } \ell/4)\\ &= \frac{\mathbb{E}\Big(\Gamma \text{ reaches height } \ell/4\,;\, \Gamma\in\mathbb{S}\,;\, e^{\Psi_{\mathbb{S}}(\Gamma)-\Psi_{\mathbb{Z}^2}(\Gamma)}\Big)}{\mathbb{E}\Big(\Gamma\in\mathbb{S}\,;\, e^{\Psi_{\mathbb{S}}(\Gamma)-\Psi_{\mathbb{Z}^2}(\Gamma)}\Big)}.\end{aligned}$$

Using Section 4.15 of [23] we get that

$$\mathbb{P}(\Gamma \text{ reaches height } \ell/4) \leqslant e^{-c\ell^2/L}$$

for some constant $c > 0$. On the other hand, (A.15) implies that

$$|\Psi_{\mathbb{Z}^2}(\Gamma) - \Psi_{\mathbb{S}}(\Gamma)| \leqslant e^{-(\beta-\beta_0)}|b\in\Gamma:\ \operatorname{dist}(b,\mathbb{S}^c) \leqslant \log(L)^2|$$

which implies

$$\mathbb{E}\Big(e^{2|\Psi_{\mathbb{Z}^2}(\Gamma)-\Psi_{\mathbb{S}}(\Gamma)|}\Big) \leqslant e^{c'\log(L)^2}$$

for some constant c', thanks to the large deviation results of Section 4.15 in [23]. Finally, thanks to (A.15) and Proposition 4.18 in [23], if $\mathcal{C}$ is the cigar-shaped region with tips at A, B defined by

$$\mathcal{C} = \Big\{(x_1, x_2) \in \mathbb{R}^2 :\ |x_2| \leqslant \Big(\frac{x_1(L-x_1)}{L}\Big)^{1/2+\kappa}\Big\},$$

$$\begin{aligned}\mathbb{E}\Big(\Gamma \in \mathbb{S}\,;\, e^{\Psi_{\mathbb{S}}(\Gamma)-\Psi_{\mathbb{Z}^2}(\Gamma)}\Big) &\geqslant \mathbb{E}\Big(\Gamma \in \mathcal{C}\,;\, e^{\Psi_{\mathbb{S}}(\Gamma)-\Psi_{\mathbb{Z}^2}(\Gamma)}\Big)\\ &\geqslant C\,\mathbb{P}(\Gamma \in \mathcal{C}) \geqslant e^{-c''\log(L)^{2/\kappa}},\end{aligned}$$

where the constant C is a deterministic lower bound on $e^{\Psi_{\mathbb{S}}(\Gamma)-\Psi_{\mathbb{Z}^2}(\Gamma)}$ for $\Gamma \in \mathcal{C}$ obtained again using (A.15). □

Appendix C.

Fix $a \in (0,1)$. Let R be the intersection between $\mathbb{Z}^2$ and a $L \times L^a$ rectangle, not necessarily parallel to the coordinate axes. Let $\Lambda \subset \mathbb{Z}^2$ be such that Λ contains R and is contained in some $2L \times 2L$ square. A subset $\mathcal{C} = \{x_1, x_2, \dots, x_k\}$ of R will be called a *spanning chain* if

(i) $d(x_i, x_{i+1}) = 1$ for all $i = 1, \dots k-1$;
(ii) $\mathcal{C}$ connects the two shorter sides of R.

For a fixed $n \geqslant 0$ let $\mathcal{F}_+$ ($\mathcal{F}_-$) be the event that there exists a spanning chain where the surface height is at least (at most) n.

Lemma C.1. *For β large enough*

$$\Pi^n_\Lambda(\mathcal{F}^c_+) \leqslant \Pi^{n,f}_\Lambda(\mathcal{F}^c_+) \leqslant e^{-cL^a}. \tag{C.1}$$

Assume moreover that $\ell(\Lambda)e^{-4\beta(n+1)} \leqslant 1$, where $\ell(\Lambda)$ is the shortest side of the smallest rectangle containing Λ. Then

$$\Pi^{n,f}_\Lambda(\mathcal{F}^c_-) \leqslant \Pi^n_\Lambda(\mathcal{F}^c_-) \leqslant e^{-cL^a}. \tag{C.2}$$

Here, as in (6.1), the field is $f = \frac{1}{L}\sum_{y\in\Lambda} f_y$.

Proof. We first observe that $\mathcal{F}_+(\mathcal{F}_-)$ is an increasing (decreasing) event and therefore the first inequalities in (C.1), (C.2) are trivial because the fields f_y are decreasing functions. Again by monotonicity $\Pi^{n,f}_\Lambda(\mathcal{F}^c_+)$ is bounded from above by the probability w.r.t. the SOS model $\hat\pi^{n,f}_\Lambda$ *without* floor. Moreover $\mathcal{F}^c_+(\mathcal{F}^c_-)$ occurs iff there exists a *-chain $\{y_1, \dots, y_n\}$ connecting the two long opposite sides of R and such that $\eta_{y_i} \leqslant n-1$ ($\eta_{y_i} \geqslant n+1$) for all i. In turn that implies the existence of a $(n-1)$-contour ($(n+1)$-contour) larger than L^a.

As in the proof of Lemma 3.7 we get that

$$\hat\pi^{n,f}_\Lambda(\gamma \text{ is a } (n-1)\text{-contour}) \leqslant e^{-\beta|\gamma|+\frac{1}{L}\sum_{x\in\Lambda_\gamma}\|f_x\|_\infty} \leqslant e^{-\frac{\beta}{2}|\gamma|}$$

where in the last inequality we used $\|f_x\|_\infty \leqslant e^{-c\beta}$ together with $|\Lambda_\gamma| \leqslant 2L|\gamma|$. Simple counting of γ finishes the proof of (C.1).

Similarly, it follows from Proposition 3.6 that

$$\Pi^n_\Lambda(\gamma \text{ is a } (n+1)\text{-contour}) \leqslant e^{-\beta|\gamma|+Ce^{-4\beta(n+1)}|\Lambda_\gamma|}$$

Isoperimetry gives $|\Lambda_\gamma| \leqslant \ell(\Lambda)|\gamma|$ which, combined with the assumption $\ell(\Lambda)e^{-4\beta(n+1)} \leqslant 1$, implies

$$\Pi^n_\Lambda(\gamma \text{ is a } (n+1)\text{-contour}) \leqslant e^{-\frac{\beta}{2}|\gamma|}$$

and the proof of (C.2) follows. □

Appendix D. Proof of inequalities (6.22) and (6.29)

Proof of Lemma 6.11. Fix $\ell > 2$. By removing the field f of (6.1) we only increase the surface so to bound the probability of the decreasing event G^+_ℓ we may work in the model $\pi^{H'}_{\Lambda_L}$, i.e. the standard SOS model on Λ_L with no field and floor/ceiling at height $0/n^+ = \log L$. If G^+_ℓ fails then for some $R \in \mathcal{R}$ we can find contours $\{(\gamma_s, h_s)\}_{s\in\mathscr{S}}$ satisfying the hypothesis of Proposition 3.6 each with $|\Lambda_{\gamma_s} \cap R| \geqslant 1$ and $h_s \geqslant H'+1$ such that $\bigcap_{s\in\mathscr{S}} \mathscr{C}_{\gamma_s,h}$ holds, and that

$$\sum_{s\in\mathscr{S}} |\Lambda_{\gamma_s} \cap R| \geqslant \ell L/2. \tag{D.1}$$

For a given ensemble of contours as above, define a sequence of subsets $W_i \subseteq \Lambda$ by

$$W_0 = \Lambda\,,$$
$$W_i = \bigcup_{s\in\mathscr{S}\,:\,h_s=H'+i} \Lambda_{\gamma_s} \qquad \text{for } i = 1,\dots,n^+ - H'\,.$$

Let $\mathcal{A}$ denote the set of all possible such collections of contours $\{(\gamma_s, h_s)\}$. For all $i \geqslant 0$, let

$$a_i = |W_i \cap R|\,.$$

Let $\mathcal{A}(\vec{a}) = \mathcal{A}(a_1, a_2, \dots, a_{n^+-H'})$ denote all collections of contours matching a given sequence of a_i's. Then (D.1) is equivalent to $\sum_{i\geqslant 1} a_i \geqslant \ell L/2$. Since R is a diagonal, $|\Lambda_{\gamma_s} \cap R| \leqslant \frac{1}{4}|\gamma_s|$ and so

$$\sum_{s\in\mathscr{S}} |\gamma_s| \geqslant 4 \sum_{i=1}^{n^+-H'} a_i\,. \tag{D.2}$$

For any $W \subseteq \Lambda$ let

$$\mathcal{B}(W) = \sum_{(\gamma'_1,\gamma'_2,\dots,\gamma'_m)} e^{-(\beta/4)|\gamma|}$$

where the sum is over all collections of edge-disjoint contours $\{\gamma'_i\}$, with pairwise disjoint interiors $\{\Lambda_{\gamma'_i}\}$ all contained in W and with $|\Lambda_{\gamma'_i} \cap R| \geqslant 1$ for all i. Any such contour must have an edge adjacent to some $v \in W \cap R$ in the dual lattice $\mathbb{Z}^{2*}$. If e is an edge in the dual lattice $\mathbb{Z}^{2*}$ then there are at most 3^n contours γ of length n containing e. Hence for large enough β,

$$\mathcal{B}(W) \leqslant \left(1 + \sum_{n=4}^{\infty} 3^n e^{-(\beta/4)n}\right)^{4|W\cap R|} \leqslant \exp(|W \cap R|)\,,$$

since each contour must contain at least one edge adjacent to some $v \in W \cap R$ in the dual lattice $\mathbb{Z}^{2*}$, there are at most $4|W \cap R|$ such edges and the contours are edge-disjoint.

Now for $\{(\gamma_s, h_s)\}_{s\in\mathscr{S}} \in \mathcal{A}(\vec{a})$ by Proposition 3.6 we have that

$$\begin{aligned} \pi^{H'}_{\Lambda_L}\left(\bigcap_{s\in\mathscr{S}} \mathscr{C}_{\gamma_s,h_s}\right) &\leqslant \exp\left(\sum_{s\in\mathscr{S}} \left(-\beta|\gamma_s| + C_0|\Lambda_{\gamma_s}|e^{-4\beta h_s}\right)\right) \\ &\leqslant \exp\left(-\frac{3}{4}\beta \sum_{s\in\mathscr{S}} |\gamma_s|\right) \end{aligned} \tag{D.3}$$

for any $\beta \geqslant C_0$ since $e^{-4\beta h_s} \leqslant e^{-4\beta(H+1)} \leqslant L^{-1}$ and $|\Lambda_{\gamma_s}| \leqslant (L/4)|\gamma_s|$ for any contour γ_s by the isoperimetric inequality in $\mathbb{Z}^2$. Substituting this expression we have that

$$\sum_{\{(\gamma_s,h_s)\}_{s\in\mathscr{S}}\in\mathcal{A}(\vec{a})} \pi_{\Lambda_L}^{H'}\Big(\bigcap_{s\in\mathscr{S}}\mathscr{C}_{\gamma_s,h_s}\Big) \leqslant \sum_{\{(\gamma_s,h_s)\}_{s\in\mathscr{S}}\in\mathcal{A}(\vec{a})} \exp\Big(-\frac{3}{4}\beta\sum_{s\in\mathscr{S}}|\gamma_s|\Big)$$
$$\leqslant \exp\Big(-2\beta\sum_{i=1}^{n^+-H'}a_i\Big)\sum_{\{(\gamma_s,h_s)\}_{s\in\mathscr{S}}\in\mathcal{A}(\vec{a})}\exp\Big(-(\beta/4)\sum_{s\in\mathscr{S}}|\gamma_s|\Big),$$

where the last inequality is by (D.2). This in turn is at most

$$\exp\Big(-2\beta\sum_{i=1}^{n^+-H'}a_i\Big)\prod_{i=1}^{n^+-H'}\mathcal{B}(W_{i-1}) \leqslant \exp\Big(-2\beta\sum_{i=1}^{n^+-H'}a_i+\sum_{i=1}^{n^+-H'}a_{i-1}\Big)$$
$$\leqslant \exp\Big(-\frac{3}{4}\beta\ell L\Big).$$

The final inequality follows for large β since $a_0 = L$. As there are at most $L^{n^+-H'} \leqslant L^{\log L}$ choices for $\vec{a} = (a_1, a_2, \dots, a_{n^+-H'})$ we have that

$$\pi_{\Lambda_L}^{H'}(G_\ell^+) \geqslant 1 - \sum_{\vec{a}}\sum_{\{(\gamma_s,h_s)\}_{s\in\mathscr{S}}\in\mathcal{A}(\vec{a})}\pi_{\Lambda_L}^{H'}\Big(\bigcap_{s\in\mathscr{S}}\mathscr{C}_{\gamma_s,h_s}\Big)$$
$$\geqslant 1 - L^{\log L}\exp\Big(-\frac{3}{4}\beta\ell L\Big)$$
$$\geqslant 1 - \exp\Big(-\frac{\beta}{2}\ell L\Big)$$

for large β, as required. □

Equation (6.22) follows similarly with a simpler proof.

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