THE THRESHOLD FOR COMBS IN RANDOM GRAPHS

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ABSTRACT. For $k \mid n$ let $\textsf{Comb}_{n,k}$ denote the tree consisting of an (n/k)-vertex path with disjoint k-vertex paths beginning at each of its vertices. An old conjecture says that for any $k = k(n)$ the threshold for the random graph $\mathcal{G}(n, p)$ to contain $\textsf{Comb}_{n,k}$ is at $p \asymp \frac{\log n}{n}$. Here we verify this for $k \leq C \log n$ with any fixed $C > 0$. In a companion paper, using very different methods, we treat the complementary range, proving the conjecture for $k \geq \kappa_0 \log n$ (with $\kappa_0 \approx 4.82$).

1. INTRODUCTION

Write $G = \mathcal{G}(n, p)$ for the usual random graph on $V := [n] := \{1, \ldots, n\}$, in which edges are present independently, each with probability p. We are interested in understanding when (i.e. for what p) G is likely to contain (a copy of) a fixed n-vertex tree T.

(Formally we may define the "threshold" for containing T to be that (unique) p for which the probability that G contains T is $1/2$. To stay closer to the usual threshold language of [7], or e.g. [10], we would need to work with a sequence $\{T_n\}$; but in any case, we will not make much use of the formal definition.)

Specifically we are interested in the following conjecture.

Conjecture 1. *For each fixed Δ there is a C such that if T is any n-vertex tree of maximum degree at most Δ, then $\mathcal{G}(n, C\frac{\log n}{n})$ w.h.p. contains T.*

(As usual "w.h.p." means with probability tending to 1 as $n \to \infty$.)

Of course for $p < \frac{\log n}{n}$ (we use log for ln), G is likely to contain isolated vertices, so Conjecture 1 says that the threshold for containing any bounded degree T is $\Theta(\frac{\log n}{n})$. This is known when T is a Hamiltonian path [6, 13], and easy when T has $\Omega(n)$ leaves (see [1, 14]). It has also been proved for "almost all" trees, even without the maximum degree requirement [4]. More recently [9], it has been shown to hold with $C = 1 + \varepsilon$ if T has $\Omega(n)$ leaves or contains a path of length $\Omega(n)$ consisting of vertices of degree 2. The best general progress to date is [14], which proves that $p \geq n^{-1+o(1)}$ suffices for all bounded degree trees, and also considers larger degrees; see this reference for some further discussion.

Conjecture 1 was proposed by the first author about twenty years ago (though stated in print only in [11], in which see also the far more general [11, Conjecture 1]), but, being a natural guess, is perhaps best considered folklore. At that early date it was also suggested that some insight might be gained by considering the case where, for some $k \mid n$, T is the tree — here denoted $\textsf{Comb}_{n,k}$

— consisting of an (n/k)-vertex path P together with disjoint k-vertex paths beginning at the vertices of P. Such trees, which have sometimes been called "combs," may be thought of as lying somewhere between the settled cases of Conjecture 1 mentioned above.

Though we have not much non-verbal evidence, this suggestion does seem to have received quite a bit of attention, but, absent any serious progress, seems not to have produced anything in print. Here and in the companion paper [12] we establish Conjecture 1 for combs.

Theorem 1.1. *There exists some fixed C such that for every n and $k \mid n$, the random graph $\mathcal{G}(n, C\frac{\log n}{n})$ w.h.p. contains a copy of $\mathsf{Comb}_{n,k}$.*

While this does not so far seem to be leading to a proof of Conjecture 1, it is plausible that our methods at least extend to any (bounded-degree) tree with $o(\sqrt{n})$ leaves.

The proof of Theorem 1.1 requires two entirely different arguments, depending on whether k is large (at least about $\log n$) or small. Here we treat small k.

Theorem 1.2. *For each D there is a K for which the following holds. If $k < D \log n$ divides n, and $v_1, \ldots, v_m$ are $m = n/k$ given (distinct) vertices, then $\mathcal{G}(n, K\frac{\log n}{n})$ w.h.p. contains m disjoint k-vertex paths rooted at the v_i's.*

This is proved in Section 2. For the easy derivation of Theorem 1.1 (for small k), we may take $G = G' \cup G''$, where G' and G'' are independent copies of, respectively, $\mathcal{G}(n, d/n)$ for a suitable constant d, and $\mathcal{G}(n, p)$. (So the p in Theorem 1.1 will be slightly larger than the one in Theorem 1.2.) Then G' w.h.p. contains a path $v_1, \ldots, v_m$ (assuming, as we may, that $k > 1$; see, e.g., [5, Chap. 8]), which, according to Theorem 1.2, we can (w.h.p.) extend to a copy of $\mathsf{Comb}_{n,k}$ using G''.

2. Proof of Theorem 1.2

For a graph H on V and disjoint $A, B \subseteq V$, we use the notation $\nabla_H(A, B) = \{xy \in E(H) : x \in A, y \in B\}$, omitting the subscript when H is the complete graph K_V. As above, we write G for $\mathcal{G}(n, p)$. Following common practice, we will sometimes pretend large numbers are integers to avoid cluttering the discussion with irrelevant floor and ceiling symbols.

Since Conjecture 1 is known to hold when T has $\Omega(n)$ leaves, we may assume k is at least any given constant. Though not really necessary, this will save us a little trouble in some places. Specifically we assume (as we may) that $D > 2$, set

$$\varepsilon = [D(10 + \log D)]^{-1}, \tag{2.1}$$

and assume $k > 2/\varepsilon$.

Set $C = 600\varepsilon^{-1}$. With apologies, we now recycle, letting $p = C\frac{\log n}{n}$, and take our random graph G to be the union of three independent copies, say G_1, G_2, G_3, of $\mathcal{G}(n, p)$. It is enough to show that G w.h.p. contains the desired paths from $v_1, \ldots, v_m$ (thus giving Theorem 1.2 with $K = 3C$).

Set $M_0 = W_0 = \{v_1, \ldots, v_m\}$ and $R = V \setminus M_0$. It is of course enough to show

Claim 2.1. *W.h.p. there is an equipartition $M_1 \cup \cdots \cup M_{k-1}$ of R such that*

$$G[M_{i-1}, M_i] \text{ admits a perfect matching for each } i \in [k-1], \tag{2.2}$$

where, for disjoint $A, B \subseteq V$, $G[A, B]$ is the bipartite graph on $A \cup B$ with edge set $\nabla_G(A, B)$.

2.1. **Algorithm.** Set $T = \lfloor mp/6 \rfloor$ and $c = mp/T$, and note that $mp = np/k > C/D > 6000$, so $T \geq 1000$. In what follows we use $N^i(x)$ (respectively $N(x)$) for neighborhood of x in G_i (resp. G). We will show (in Section 2.2) that the following procedure w.h.p. produces a partition as in Claim 2.1.

First step: Let $\alpha \leq 1$ be a constant to be specified later and $Z = \{x \in R : |N^1(x) \cap W_0| < T\}$. Let $W_1, \ldots, W_{k-1}$ be disjoint random subsets of R given by

$$\mathbb{P}(x \in W_i) = \alpha/k \begin{cases} \forall i \in [k-1] & \text{if } x \notin Z, \\ \forall i \in \{2, \ldots, k-1\} & \text{if } x \in Z, \end{cases}$$

these choices independent for different vertices x. (Thus $\mathbb{P}(x \notin \cup W_i)$ is $1 - \alpha$ or $1 - \alpha(1 - 1/k)$, as the case may be.) Set $W = \cup_{i=0}^{k-1} W_i$. The W_i's are our initial installments on the M_i's, to be augmented in the next two steps. (We won't bother with formal notation for the evolving M_i's.)

It will be helpful to define $L(i) = \{i - 1, i + 1\} \cap \{0, \ldots, k - 1\}$ for $0 \leq i \leq k - 1$. For $i \in [k - 1]$, set

$$B_i = \{x \in R \setminus W : \exists j \in L(i), \; |N^1(x) \cap W_j| < T\};$$

these vertices will be barred from M_i. (In particular $B_1 \supseteq Z$.)

Repair phase: For $i \in \{0, \ldots, k - 1\}$ and $j \in L(i)$, let

$$X_{ij} = \{x \in W_i : |N^1(x) \cap W_j| < T\}.$$

(In particular $X_{10} = W_1 \cap Z = \emptyset$.) We repair the X_{ij}'s in some arbitrary order. *Repairing* $X_{ij} = \{x_1, \ldots, x_s\}$ means that for $r = 1, \ldots, s$ we choose (again, arbitrarily) T available vertices from $N^2(x_r)$ and add them to M_j, where a vertex is *un*available if it belongs to B_j or has already been assigned to one of the M_u's. Note that the set of edges — say, E^2 — used in these "repairs" (i.e. edges from x_r to the chosen vertices in $N^2(x_r)$) is a (star-)forest.

Filling in: Assign the as yet unassigned vertices to the M_i's so that

$$\text{for all } i, \; |M_i| = m \text{ and } M_i \cap B_i = \emptyset. \tag{2.3}$$

The main point in all this is that, since vertices of B_i are barred from M_i in the repair and filling in phases, at the end of each of these phases, we have $|N_G(x) \cap W_j| \geq T$ for each $x \in M_i$ and $j \in L(i)$.

2.2. **Analysis.** We want to show that w.h.p. (i) the above procedure runs to completion and (ii) the M_i's produced satisfy (2.2). (It may be worth observing that G_3, which plays no role in (i), is needed for (ii).) Recalling that ε was specified in (2.1), set

$$\alpha = 1/3, \;\; \gamma = (1 - 3\varepsilon)\alpha, \;\; \beta = \frac{(c\gamma - 1)^2}{4c\gamma}, \;\; \text{and} \;\; q = 2e^{-\beta T}.$$

We first need some routine observations.

Proposition 2.2. *The objects produced by the first step above w.h.p. satisfy*

(a) $|Z| \leq \varepsilon n$;

(b) $|W_i| \in (\gamma m, (1 + \varepsilon)\alpha m) \; \forall i \in [k - 1]$;

(c) $|B_i| < \varepsilon n \; \forall i \in [k - 1]$;

(d) *no vertex is in more than εk of the B_i's;*

(e) $|X_{ij}| < 2mq + \log n \quad \forall i \in \{0, \ldots, k - 1\}$ *and* $j \in L(i)$.

Of course (c) contains (a), but we state (a) first since it's needed for (b), which in turn is needed for (c).

Note that the events in Proposition 2.2 depend only on G_1 and the W_i's. In fact it will be helpful to conserve some of this information: for $x \in V$ and $i \in \{0, \ldots, k - 1\}$, let $\zeta(i, x)$ be the indicator of the event $\{|N^1(x) \cap W_i| \geq T\}$. Then $\{x \in Z\} = \{\zeta(0, x) = 0\}$ $(x \in R)$ and, once we have the W_i's, the remaining assertions in the proposition are functions of the $\zeta(i, x)$'s.

There is nothing delicate about Proposition 2.2, and we aim for simple rather than optimal arithmetic. The following Bernstein/Chernoff-type bound (for which see e.g. [3, Lemma 8.2]) will be sufficient for our large deviation purposes. (We use $B(m, \rho)$ for a r.v. with the binomial distribution $\text{Bin}(m, \rho)$.)

Lemma 2.3. *For any m, ρ and $t > 0$,*

$$\left.\begin{array}{l} \mathbb{P}(B(m, \rho) > m\rho + t) \\ \mathbb{P}(B(m, \rho) < m\rho - t) \end{array}\right\} < \exp[-\tfrac{1}{4} \min\{t, t^2/m\rho\}].$$

Proof of Proposition 2.2. (a) For $x \in R$, we have, using Lemma 2.3,

$$\begin{aligned} \mathbb{P}(x \in Z) &= \mathbb{P}(B(m, p) < T) \\ &= \mathbb{P}(B(m, p) < mp - (c - 1)T) \\ &< \exp[-\tfrac{(c-1)^2}{4c} T] < q. \end{aligned}$$

Thus, writing "$\succ$" for stochastic domination, we have $|Z| \prec B(n, q)$, whence, using Lemma 2.3 and $\varepsilon > 2q$, $\mathbb{P}(|Z| > \varepsilon n) < \exp[-(\varepsilon - q)n/4]$.

(b) Given Z satisfying (a) we have, for each i, $|W_i| \sim \text{Bin}(n_i, \alpha/k)$, where $n_1 = n - m - |Z|$ and $n_i = n - m$ if $i \geq 2$. In particular (for each i), $n_i \in ((1 - 2\varepsilon)n, n)$ (note $m < \varepsilon n$ because of our lower bound on k), and

$$\begin{aligned} \mathbb{P}(|W_i| \notin (\gamma m, (1 + \varepsilon)\alpha m)) &\leq \mathbb{P}(|W_i| \notin ((1 - \varepsilon)\alpha n_i/k, (1 + \varepsilon)\alpha n_i/k)) \\ &< 2\exp[-\varepsilon^2 \alpha m/4]. \end{aligned}$$

(c) and (d). Condition on values of Z and the W_i's satisfying (a) and (b) — note this uses the values $\zeta(0, x)$ $(x \in R)$ but no other information from G_1 — and write $\mathbb{P}'$ for the corresponding conditional probabilities. (We may of course think of exposing just the edges of G_1 incident with W_0 to determine $\mathbb{P}'$.)

For $x \in R \setminus W$ and $i \in [k-1]$, again using Lemma 2.3, we have

$$
\begin{aligned}
\mathbb{P}'(x \in B_i) \quad &< \quad 2\mathbb{P}(B(\gamma m, p) < T) \\
&= \quad 2\mathbb{P}(B(\gamma m, p) < \gamma m p - (\gamma c - 1)T) \quad < \quad q, \quad (2.4)
\end{aligned}
$$

unless $i = 1$ and $x \in Z$, in which case x is automatically in B_1. (If $i = 1$ and $x \notin Z$, the 2's in (2.4) are unnecessary.)

Using (2.4) and independence of the events $\{x \in B_i\}$ ($x \in R \setminus W, i \in [k-1]$), we have (i) $|B_1 \setminus Z|, |B_2|, \ldots, |B_{k-1}| \prec B(n, q)$, so that (c) holds with probability at least $1 - k \exp[-(\varepsilon - q)n/4]$, and (ii) for any $x \in R \setminus W$,

$$
\begin{aligned}
\mathbb{P}(|\{i \geq 2 : x \in B_i\}| \geq \lceil \varepsilon k \rceil - 1) \quad &< \quad \binom{k}{\lceil \varepsilon k \rceil - 1} q^{\varepsilon k - 1} \\
&< \quad (e/\varepsilon)^{\varepsilon k} \exp[-\tfrac{\beta C \varepsilon}{2c} \log n] \\
&< \quad \exp[(D \log \tfrac{e}{\varepsilon} - \tfrac{\beta C}{2c})\varepsilon \log n] = o(1/n).
\end{aligned}
$$

Here we used $\binom{k}{r} \leq (ek/r)^r \leq (e/\varepsilon)^{\varepsilon k}$, the latter valid for $r \leq \varepsilon k$; $\varepsilon k - 1 > \varepsilon k/2$; $T = mp/c = C \log n/(ck)$; $k < D \log n$; and, for the $o(1/n)$, the easily verified $\beta C/(2c) - D \log(e/\varepsilon) > 2/\varepsilon$.)

(e) We retain the conditioning and notation $\mathbb{P}'$ of (c). We assume first that $(i, j) \neq (0, 1)$ (and, since $X_{10} = \emptyset$, may also assume $(i, j) \neq (1, 0)$). For $x \in W_i$ we have, as in (2.4),

$$
\mathbb{P}'(x \in X_{ij}) < \mathbb{P}(B(\gamma m, p) < T) < q, \quad (2.5)
$$

whence $|X_{ij}| \prec B(m, q)$ and (again using Lemma 2.3)

$$
\mathbb{P}'(|X_{ij}| \geq 2mq + \log n) < \exp[-(mq + \log n)/4] < n^{-1/4} = o(1/k).
$$

For $(i, j) = (0, 1)$ the preceding argument is not quite applicable, since conditioning on $A := \{W_1 \cap Z = \emptyset\} = \{\zeta(0, x) = 1 \ \forall x \in W_1\}$ introduces dependencies among the edges joining W_0 and W_1. But since A is an increasing event, Harris' Inequality [8] says that this conditioning does not increase the probability of the decreasing event $\{|X_{01}| \geq 2mq + \log n\}$; so the argument in the preceding paragraph does imply $\mathbb{P}'(|X_{01}| \geq 2mq + \log n) = o(1/k)$. (Of course this detail could also be dealt with by simply choosing additional random edges between W_0 and W_1.) ∎

Write Q for the intersection of the events in (a)–(e), and S for the event that our process does not get stuck — that is, there *are* T available vertices whenever the repair phase requires them and there *is* a way to complete the M_i's in the filling in phase — and the M_i's it produces satisfy (2.2). We have

$$
\mathbb{P}(\overline{S}) \leq \mathbb{P}(\overline{Q}) + \mathbb{P}(\overline{S} \mid Q) = o(1) + \mathbb{P}(\overline{S} \mid Q),
$$

so just need $\mathbb{P}(\overline{S} \mid Q) = o(1)$.

The first part of S — that the process doesn't get stuck — is easy. First, given Q, the number of available vertices at any repair step (at x say) is at least

$$
n - m - (|W| + \max_i |B_i| + T \sum |X_{ij}|) > n/2. \quad (2.6)
$$

To see this notice that, since there are at most $2k$ terms in the sum, we may bound the third term in brackets using (e) and

$$Tq = 2Te^{-\beta T} \leq 2\frac{C}{Dc}\exp[-\frac{C\beta}{Dc}] < \varepsilon$$

(say). Here the first inequality is gotten by noting that $xe^{-\beta x}$ is decreasing on $x > 1/\beta$ and that $T = mp/c \geq C/(Dc)$. The second may be rewritten as

$$\tfrac{1200}{c}\exp[-\tfrac{600\beta}{\varepsilon Dc}] < \varepsilon^2 D\,,$$

which, since $c \geq 6$ and $600\beta/c > 5$ (say), follows from the easily verified

$$200\exp[-5(10 + \log D)] < D^{-1}(10 + \log D)^{-2}.$$

We conclude that the probability that x has fewer than T available neighbors in G_2 is at most $\mathbb{P}(B(n/2, p) < T) = o(1/n)$, so that the repair phase w.h.p. finishes successfully.

   Second, to say that the filling in step w.h.p. finishes successfully, it's enough to show that Q implies the existence of an assignment of M_i's satisfying (2.3). This is a standard type of application of Hall's Theorem, briefly as follows. For $i \in [k - 1]$, write W_i^* for the set of vertices assigned to M_i through the end of the repair phase, and set $W^* = \cup W_i^*$, $r_i = m - |W_i^*|$ and $r = \sum r_i = |R \setminus W^*|$. A set of M_i's with (2.3) is equivalent to a perfect matching in the bipartite graph Γ on the vertex set $\{v_{ij} : i \in [k - 1], j \in [r_i]\} \cup (R \setminus W^*)$ with $v_{ij} \sim x$ iff $x \notin B_i$. Then: the common size of the two sides of the bipartition is $r \in (n/2, n)$ (see (2.6) for the lower bound); for degrees in Γ we have $d(v_{ij}) = |R \setminus (W^* \cup B_i)| > n/2 > r/2$ (again see (2.6)) and, using (d), $d(x) = r - \sum\{r_i : x \in B_i\} > r - \varepsilon km > r/2$; and it follows easily from Hall's Theorem that a bipartite graph with r vertices in each part of the bipartition and all degrees at least $r/2$ admits a perfect matching.

   We are left with the more interesting part of S, the assertion that (2.2) holds w.h.p. given Q. Say A is a *violator of type* (i, j, a) if $A \subseteq M_i$, $|A| = a$, and $|N_j(A)| < a$, where $N_j(A) = N(A) \cap M_j$ (and $N(A) = \cup_{x \in A} N(x)$). By Hall's Theorem it is enough to show the following (given Q).

Claim 2.4. *W.h.p. there is no violator of type* (i, j, a) *for any* $a \in \{1, \ldots, \lceil m/2 \rceil\}$, $i \in \{0, \ldots, k-1\}$ *and* $j \in L(i)$

(since if A is a violator of type (i, j, a) for some $a > \lceil m/2 \rceil$, then $M_j \setminus N_j(A)$ contains a violator of type $(j, i, \lceil m/2 \rceil)$).

Proof. Fix i, j as in the claim and set $\vartheta = (ce)^{-2}$. We consider the cases $a \leq \vartheta m$ and $a > \vartheta m$ separately, beginning with the former.

   Let E^1 be the set of edges of G_1 that meet W, and recall E^2 is the set of edges of G_2 that are actually used in the repair phase. If A is a violator of type (i, j, a), then there is some a-subset B of M_j containing $N_j(A)$. (We could, of course, require $|B| < a$.) The algorithm arranges that each vertex of M_i is joined by $E^1 \cup E^2$ to at least T vertices of M_j (we actually make no use of $(E(G_1) \setminus (E^1)) \cup (E(G_2) \setminus E^2)$), whence

$$|(E^1 \cup E^2) \cap \nabla(A, B)| = |(E^1 \cup E^2) \cap \nabla(A, M_j)| \geq aT\,,$$

while (since E^2 is a forest) $|E^2 \cap \nabla(A, B)| < 2a$; so

$$|E^1 \cap \nabla(A, B)| > a(T - 2).$$

Thus the probability of a violator of type (i, j, a) is at most

$$\sum_{A,B} \mathbb{P}(Q_a(A, B)), \tag{2.7}$$

where $Q_a(A, B)$ is the event $\{A \subseteq M_i, B \subseteq M_j, |E^1 \cap \nabla(A, B)| \geq a(T - 2)\}$ if $|A| = |B| = a$ and $Q_a(A, B) = \emptyset$ otherwise, and the sum is over $A, B \subseteq V$. (Of course if $\{i, j\} \neq \{0, 1\}$ then the only nonzero summands are those with A, B disjoint a-subsets of R, and, for example, when $(i, j) = (0, 1)$ we are only interested in pairs with $A \subseteq W_0$ and $B \subseteq R$ (and $|A| = |B| = a$).) Note that, summing only over a-subsets A, B of V, we have

$$\sum_{A,B} \mathbb{P}(A \subseteq M_i, B \subseteq M_j) = \binom{m}{a}^2 \tag{2.8}$$

(since the r.v. $\sum_{A,B} \mathbf{1}_{\{A \subseteq M_i, B \subseteq M_j\}}$ is actually the constant $\binom{m}{a}^2$; of course by symmetry the summand in (2.8) is the same for all (A, B) of interest, but we don't need this).

On the other hand, we will show (provided the conditioning event is not vacuous)

$$\mathbb{P}(Q_a(A, B) \mid A \subseteq M_i, B \subseteq M_j) < (1 - q)^{-2a} \binom{a^2}{a(T - 2)} p^{a(T-2)}. \tag{2.9}$$

Given this we just need a little arithmetic: the combination of (2.8) and (2.9) yields

$$
\begin{aligned}
\sum_{A,B} \mathbb{P}(Q_a(A, B)) \ &\leq \ \binom{m}{a}^2 (1 - q)^{-2a} \binom{a^2}{a(T - 2)} p^{a(T-2)} \\
&\leq \ \left[(1 - q)^{-2} \left(\frac{em}{a} \right)^2 \left(\frac{eap}{T - 2} \right)^{T-2} \right]^a \\
&= \ \left[\left(\frac{e}{1 - q} \right)^2 \left(\frac{a}{m} \right)^{T-4} \left(\frac{emp}{T - 2} \right)^{T-2} \right]^a \\
&< \ \left[(ce)^T \left(\frac{a}{m} \right)^{T-4} \right]^a \tag{2.10}
\end{aligned}
$$

(say), which easily implies

$$\sum_{a=1}^{\lfloor \vartheta m \rfloor} \sum_{A,B} \mathbb{P}(Q_a(A, B)) < O(1/m)^{T-4}. \tag{2.11}$$

It remains to prove (2.9). Here it is helpful to think of our procedure as choosing

(i) $\zeta(0, x)$ for $x \in R$, thus specifying Z;

(ii) $W_1, \ldots, W_{k-1}$;

(iii) $\zeta(i, x)$ for $i \in [k - 1]$ and $x \in V \setminus W =: Y$, thus specifying the B_i's

(and then continuing). It is then evident that the only information from $E(G_1)$ with any bearing on our choices of the sets W_l and $M_l \setminus W_l$ is that in (i) and (iii); in particular, we have the following.

Observation 2.5. *The pair (M_i, M_j), set $E^1 \cap \nabla(W_i \cup W_j, Y)$ and indicators $\mathbf{1}_{\{xy \in E^1\}}$ for $(x, y) \in W_i \times W_j$ are conditionally (mutually) independent given W_i, W_j and the values of $\zeta(i, x)$ and $\zeta(j, x)$ for $x \in Y$.*

Suppose now that we're given W_i, W_j, M_i, M_j with $A \subseteq M_i$ and $B \subseteq M_j$. For a set X we use $B(X, p)$ for the distribution on the power set of X that assigns $U \subseteq X$ probability $p^{|U|}(1-p)^{|X \setminus U|}$.

We assume first that we are not in one of the slightly special cases with $\{i, j\} = \{0, 1\}$. According to Observation 2.5, the sets $E^1 \cap \nabla(x, W_i)$ and $E^1 \cap \nabla(x, W_j)$ ($x \in Y$) and the indicators $\mathbf{1}_{\{xy \in E^1\}}$ ($x \in W_i, y \in W_j$) are mutually independent. Each of the indicators is Bernoulli with mean p; each $E^1 \cap \nabla(x, W_i)$ is distributed as $\mathbf{F} := B(\nabla(x, W_i), p)$ conditioned on $\{|\mathbf{F}| \geq T\}$, an event of probability at least $\mathbb{P}(\mathrm{Bin}(\gamma m, p) \geq T) > 1 - q$ (see Proposition 2.2(b) and (2.4)); and similarly for the $E^1 \cap \nabla(x, W_j)$'s. Thus we can bound the probability of any event determined by $E^1 \cap \nabla(A, B)$, by computing its probability assuming all edges occur independently with probability p, and then multiplying by $(1 - q)^{-|A \setminus W_i| + |B \setminus W_j|} \leq (1 - q)^{-2a}$. This gives (2.9):

$$
\begin{aligned}
\mathbb{P}(Q_a(A, B) \mid A \subseteq M_i, B \subseteq M_j) &< (1 - q)^{-2a} \mathbb{P}(\mathrm{Bin}(a^2, p) \geq a(T - 2)) \\
&< (1 - q)^{-2a} \binom{a^2}{\lfloor a(T - 2) \rfloor} p^{a(T-2)}.
\end{aligned}
$$

When $\{i, j\} = \{0, 1\}$, $Q_a(A, B)$ is determined by the sets $E^1 \cap \nabla(x, W_0)$, for $x \in A$ if $i = 1$ and $x \in B$ if $i = 0$. Recalling that the choice of M_1 depends only on $\zeta(0, x)$ for $x \in R$, these sets are independent, each distributed as $\mathbf{F} := B(\nabla(x, W_0), p)$ conditioned on $\{|\mathbf{F}| \geq T\}$, an event of probability at least $\mathbb{P}(B(m, p) \geq T) > 1 - q$, and (2.9) follows as before. (In this case $(1 - q)^{-2a}$ could be replaced by $(1 - q)^{-a}$.) ∎

For the simpler analysis when $a > \vartheta m$ (and $a \leq \lceil m/2 \rceil$), we just use G_3. Here a violator A of type (i, j, a) satisfies $\nabla(A, B) = \emptyset$ for some $B \subseteq M_j$ of size $\lceil m/2 \rceil$; so the probability of such a violator is less than

$$
\sum_{A, B} \mathbb{P}(A \subseteq M_i, B \subseteq M_j, E(G_3) \cap \nabla(A, B) = \emptyset) < 4^m (1 - p)^{am/2} = o(1),
$$

where the sum is over disjoint $A, B \subseteq V$ (but really, for example, over $A, B \subseteq R$ unless $\{i, j\} = \{0, 1\}$) with $|A| = a$ and $|B| = \lceil m/2 \rceil$, and we used

$$
\sum_{A, B} \mathbb{P}(A \subseteq M_i, B \subseteq M_j) = \binom{m}{a} \binom{m}{\lceil m/2 \rceil},
$$

$$
\mathbb{P}(E(G_3) \cap \nabla(A, B) = \emptyset \mid A \subseteq M_i, B \subseteq M_j) \leq (1 - p)^{am/2}
$$

(of course here G_3 is actually independent of the conditioning), and (recalling $k < D \log n$) $\vartheta m p / 2 \geq \vartheta C / (2D) > 2 \log 4$. ∎

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