Homework 2

Exercise 1: Risk neutral measure in a trinomial setting

Consider a trinomial tree in a one period economy:

$$\begin{tikzcd}
 & S_u \\
S_0 \arrow[ur, "p_1"] \arrow[r, "p_2"'] \arrow[dr, "p_3"'] & S_m \\
 & S_d
\end{tikzcd}$$

for some $S_u \geq S_m \geq S_d$ and $p_i \geq 0$, $\sum_i p_i = 1$. Assume that there is an interest rate $r \geq 0$ per period (that is 1\$ at t yields $(1+r)$\$ at $t+1$), and that $S_d < (1+r)S_0 < S_u$.

(a) Does there exist a risk neutral measure? Is it unique? Exhibit one example if it exists or explain why there are no examples.

(b) Assume you have an option with payoff V_u, V_m, V_d in the corresponding situations. Is there a unique price? If so write it down. If not, what are the possible prices (give an upper and lower bound)?

The upper bound is called the seller's price, and the lower bound the buyer's price.

Exercise 2: Pricing with the Black-Scholes formula and beyond

Assume that we are in the Black-Scholes setting, that is the stock price is given in the risk neutral measure by:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

with some constant interest rate $r > 0$ and volatility $\sigma > 0$, and W_t a Brownian motion in the risk neutral measure. Assume that today's price $S_t = s > 0$.

We saw in class that the price at time t of a European Call option with strike K and maturity T, that is an option with payoff $(S_T - K)$ at time T, is given by:

$$C(t, s; K, T, \sigma, r) = sN(d_+) - Ke^{-r\tau}N(d_-)$$

with $\tau = T - t$, $N(z) = \displaystyle\int_{-\infty}^{z} \frac{\exp(-y^2/2)}{\sqrt{2\pi}} dy$ and

$$d_- = d_+ - \sigma\sqrt{\tau}, \quad d_+ = \frac{1}{\sigma\sqrt{\tau}}\left(\log(s/K) + \left(r + \frac{1}{2}\sigma^2\right)\tau\right)$$

Put-Call parity

(a) A forward contract with strike K and maturity T pays $S_T - K$ at time T. That is, we agree exchanging the stock at the price K, at time T. The price today of such a contract is given by the risk neutral formula:

$$F(t, s; K, T, r) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)|S_t = s\right]$$

give a one line justification that proves:

$$F(t, s; K, T, r) = s - e^{-r(T-t)}K$$

(b) We could do a computation similar to the one for the Call to get the price of a European put option, that is for an option that pays $(K - S_T)_+$ at time T.

Instead of computing ugly integrals, use the fact that for any $x, K \in \mathbb{R}$:

$$x - K = (x - K)_+ - (K - x)_+$$

to compute the price at time t of a European put option with strike K and maturity T.

Payoff decomposition (finite case)

(c) Let's extend this decomposition to other options; Using the Black-Scholes formula for the price of a European Call, give an analytical formula for the price of a Bull call spread which payoff is given by

$$V(S_T) = \begin{cases} B, \text{ if } S_T > B \\ \frac{B+A}{B-A}S_T - \frac{2AB}{B-A}, \text{ if } S_T \in [A, B] \\ -A, \text{ if } S_T < A \end{cases}$$

for some $0 < A < B$. *Hint: Can this payoff be replicated with a combination of calls?*

(d) Do the same for a Butterfly spread which payoff is given by

$$V(S_T) = \begin{cases} 0, \text{ if } S_T < K - \delta \\ \frac{1}{\delta}(S_T - (K - \delta)), \text{ if } S_T \in [K - \delta, K) \\ -\frac{1}{\delta}(S_T - (K + \delta)), \text{ if } S_T \in [K, K + \delta] \\ 0, \text{ if } S_T > K + \delta \end{cases}$$

for some $K, \delta > 0$.

(e) Explain the practical advantages of using such decompositions, as opposed to pricing directly using a PDE method or Monte-Carlo.

Hint: What are the implications of these decompositions in terms of replication/hedging? Would you rather use the Δ *from the PDE method?*

Payoff decomposition (infinite case)

Some payoffs aren't a linear combination of calls, puts, forwards. etc., and hence we can't use a simple decomposition and linearity of expectation. We can still however use the butterfly spreads to approximate them by such a linear combination.

Set $v(x; K) = (x - K)_+, \forall x, K \in \mathbb{R}$.

(e) Draw the shape of the function V defined in question (d), that is

$$V(x; K, \delta) = \frac{v(x; K + \delta) - 2v(x; K) + v(x; K - \delta)}{\delta}$$

for $\delta \geq 0, K \in \mathbb{R}$.

(f) Let f be a C^0 function on an interval $[a,b]$. Show that the function f_N defined on $[a,b]$ for $N \in \mathbb{N}^*$ by:

$$f_N(x) = \sum_{i=0}^{N} f(a+i\delta_N) V(x; a+i\delta_N, \delta_N)$$

for $\delta_N = \frac{b-a}{N}$, converges uniformly to f as $N \to +\infty$.

Hint: What kind of approximation of f is f_N?

(g) Explain how you can approximately price an option with an arbitrary payoff using Butterfly spreads (or calls).

Note that you don't theoretically need the Black-Scholes formula to price this way; 'just' observe the call prices on the market. On a practical side, you may not have all prices available for all strikes, and thus need to rely on an interpolation. This interpolation needs to be carefully implemented, otherwise you might introduce arbitrage opportunities.

Density of S_T in the risk neutral measure

Let's examine closely the formula in question (f); we can formally write it as

$$\begin{aligned} f(x) &\approx \sum_{i=0}^{N} f(a+i\delta_N) \frac{v(x; a+(i+1)\delta_N) - 2v(x; a+i\delta_N) + v(x; a+(i-1)\delta_N)}{\delta_N^2} \delta_N \\ &= \int_a^b f(K) \frac{\partial^2}{\partial K^2} v(x; K) dK \end{aligned}$$

where the last approximation comes from the Riemann sum definition of an integral. Using linearity of the expectation, this would imply that the price P at time t of an option with payoff $f(S_T)$ is given by

$$P(t, S_t = s) = \int f(K) \left[\frac{\partial^2}{\partial K^2} C(t, s; K, T, \sigma, r) \right] dK$$

This is to be contrasted with the risk neutral formula;

$$P(t, S_t = s) = \mathbb{E}_{\mathbb{Q}} \left[e^{-r(T-t)} f(S_T) | S_t = s \right] = \int f(y) [e^{-r(T-t)} p(T, y; t, s)] dy$$

where $p(T, y; t, s)$ is the density at time T of S_T given that $S_t = s$.

This suggests that

$$p(T, y; t, s) = e^{r(T-t)} \frac{\partial^2}{\partial K^2} C(t, s; K, T, \sigma, r) \quad (\star)$$

The issue with this reasoning is that $v(x; K)$ is not differentiable at $K = x$.

There is however a simple way to show that:

(h) Use the risk neutral formula on a European call:

$$C(t, s; K, T, \sigma, r) = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} (S_T - K)_+ | S_t = s] = \int (y-K)_+ p(T, y; t, s) dy$$

to show $(\star)$.

Hint: Take $\partial/\partial K$ twice.

We just showed that regardless of the model used (Black-Scholes or not), we can always use the prices of European calls to deduce the density of the stock price in the risk neutral measure. This allows us to deduce prices that are consistent with these options, that is any inconsistency can be taken advantage of using a static hedge of Calls. Of course the portfolio needs to be infinite, and hence an exact hedge isn't always feasable.

Exercise 3: Local Volatility Model

We saw in class that assuming a Black-Scholes dynamic for the stock price isn't a realistic model for option pricing; in the case of European Calls for example, the volatility σ would have to depend on the strike K. We labeled this volatility 'implied volatility', as it is the one consistent with the observed prices.

Can we devise a consistent model for the stock price dynamics that would recover the observed prices for any strike?

This is what the local volatility (Dupire 1994) model is about: Let $C(T, K)$ be the observed price at time t of European Calls of strike K and maturity T. Assume that C is a $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+)$ function, that is continuously differentiable w.r.t. T and twice continuously differentiable w.r.t. K.

The goal is to show that there exists a unique function $\sigma_{LV}(t, x)$ such that the stock price defined by the Markovian SDE:

$$dS_t = r_t S_t dt + \sigma_{LV}(t, S_t) dW_t$$

for deterministic r_t and W_t a Brownian motion under some measure $\mathbb{Q}$, satisfies the risk neutral pricing formula:

$$C(T, K) = \mathbb{E}_{\mathbb{Q}}\left[\frac{D_T}{D_t}(S_T - K)_+ \middle| S_t = s\right] \quad (\blacktriangle)$$

for $D_t = e^{-\int_0^t r_s ds}$.

The exercise above shows that having a consistent model for European calls yields a consistent model for arbitrary vanilla options (with payoff of the type $f(S_T)$).

(a) Assume such a σ_{LV} exists. Let's find some necessary conditions; write the forward PDE satisfied by the density $p(T, S; t, s)$ (or $p(T, S)$ in short) of the stock price S_T under such a model.

(b) Write ($\blacktriangle$) in terms of p. Take ∂_T (assuming you can exchange integral and derivative) and use the forward equation to replace $\partial_T p$ by its spatial derivatives.

(c) Integrate by parts to get rid of any spatial derivative on p. You can assume that the boundary terms vanish [1].

(d) Similarly, compute $\partial_K C$ and $\partial_{KK} C$ from ($\blacktriangle$), and use them to replace all terms involving p in your answer to (c).

(e) Deduce that one necessarily has:

$$\sigma_{LV}^2(T, K) = \frac{\partial_T C(T, K) + r_T K \partial_K C(T, K)}{\frac{1}{2} K^2 \partial_{KK} C(T, K)}$$

One can show that the above formula gives a sufficient condition on the evolution on the stock dynamics to be consistent with Call options' prices.

Note that σ_{LV} implicitly depends on t and S_t (from $C(T, K)$).

[1] let's assume that this SDE yields densities in the Schwartz class, that is densities (or its derivatives) which decay faster than any polynomials