Continuous Time Finance

Homework 1: Review

In all exercises, $(\Omega, \mathcal{F}, \mathbb{P})$ represents a probability space, $(W_t)_{t\geq 0}$ a standard Brownian motion on that probability space, and $(\mathcal{F}_t)_{t\geq 0}$ the filtration it generates.

Exercise 1: Brownian motion definitions

Consider the following two definitions of Brownian Motion:

Definition 1:

(i) $W_0 = 0$ $\mathbb{P}$-almost surely

(ii) $\forall 0 \leq r < s \leq t < u$, $W_u - W_t$ is independent of $W_s - W_r$

(iii) $\forall s \neq t$, $W_t - W_s \sim \mathcal{N}(0, |t-s|)$

(iv) $t \mapsto W_t$ is continuous $\mathbb{P}$-almost surely

Definition 2:

(i) $W_0 = 0$ $\mathbb{P}$-almost surely

(ii) $\forall n \in \mathbb{N}^*$, and $0 \leq t_1 < \cdots < t_n$, $(W_{t_1}, \cdots, W_{t_n})$ is a Gaussian vector of mean $0 \in \mathbb{R}^n$ and covariance matrix $\Sigma = [\min(t_i, t_j)]_{i,j=1,\cdots,n} \in \mathbb{R}^{n\times n}$

(iii) $t \mapsto W_t$ is continuous $\mathbb{P}$-almost surely

Show that both definitions are equivalent.

Exercise 2: Reflection principle for Brownian motion

Define the stochastic process for $t \in \mathbb{R}_+$:

$$M_t = \max_{0\leq s\leq t} W_s$$

which is the running maximum of a Brownian motion. Also define the random variable for $b \in \mathbb{R}$:

$$\tau_b = \inf\{t \geq 0 : W_t = b\}$$

which gives the first time W reaches the level b.

The goal is to derive their distribution. To do so, we will first compute:

$$F(a, b) = \mathbb{P}(W_t \leq a, M_t \geq b)$$

by the so called 'Reflection principle'.

(a) Let $0 \leq a \leq b$, $t \in \mathbb{R}_+$ and define the stochastic process for $s \in [0,t]$:

$$\tilde{W}_s = \begin{cases} W_s, & \text{if } s \leq \tau_b \\ 2b - W_s, & \text{if } s \geq \tau_b \end{cases}$$

Plot a Brownian path W on $[0,t]$, satisfying: $\tau_b < t$ and $W_t \leq a$. Plot the corresponding path (i.e. the same 'ω') for $\tilde{W}$ on $[0,t]$. In what interval does $\tilde{W}_t$ end up?

(b) Let's admit that $\tilde{W}$ is still a Brownian motion[1], and hence

$$F(a,b) = \mathbb{P}\left(\tilde{W}_t \leq a, \max_{0 \leq s \leq t} \tilde{W}_s \geq b\right)$$

By using the definition of $\tilde{W}$, and by noting that:

$$\left\{\omega \in \Omega : \max_{0 \leq s \leq t} \tilde{W}_s \geq b\right\} = \{\omega \in \Omega : \tau_b \leq t\},$$

show that:

$$F(a,b) = \mathbb{P}(W_t \geq 2b - a)$$

(c) Deduce $\mathbb{P}(M_t \geq b)$ (Hint[2]), $\mathbb{P}(\tau_b \leq t)$, the densities of M_t, τ_b as well as the joint distribution of (W_t, M_t).

Exercise 3: Time independent boundary value problems

Let $D = [a,b]$ and consider the stochastic process:

$$dX_t = \alpha(X_t)dt + \beta(X_t)dW_t$$

Note that α, β are deterministic functions that do not depend on time. Define

$$u(x) = \mathbb{E}\left[\int_0^{\tau_x} f(X_s)ds + g(X_{\tau_x}) \middle| X_0 = x\right] \tag{1}$$

for $x \in D$, where

$$\tau_x = \{\inf t \geq 0 : X_t \notin D\}$$

Note that τ_x depends on x due to the starting point $X_0 = x$. f, g are deterministic functions, that represent respectively a *running payoff* and a *final time payoff*.

In other words, we are playing a game where we receive (or pay) $f(X_t)dt$ for each unit of time dt as long as X_t remains in D. As soon as X_t exits D, we get (or pay) $g(X_t)$. $u(x)$ represents our expected payoff from this game.

The goal is to show that u solves the ODE (becomes a PDE if $x \in \mathbb{R}^n$):

$$\begin{cases} \alpha(x)\dfrac{d}{dx}u(x) + \dfrac{\beta^2(x)}{2}\dfrac{d^2}{dx^2}u(x) + f(x) = 0, & x \in D \\ u(a) = g(a), \quad u(b) = g(b) \end{cases}$$

(a) Apply Ito's lemma to $u(X_t)$ and integrate both sides of the equation between 0 and τ_x.

(b) Assume that u does indeed solves the ODE above. Deduce that

$$u(x) = \mathbb{E}\left[\int_0^{\tau_x} f(X_s)ds + g(X_{\tau_x}) \middle| X_0 = x\right]$$

[1] a consequence of the independence of W_{τ_b} and $W_s - W_{\tau_b}$ for $s \geq \tau_b$, and of $u \mapsto W_{\tau_b+u} - W_{\tau_b}$ being a Brownian Motion

[2] *Hint:* $\mathbb{P}(M_t \geq b) = \mathbb{P}(M_t \geq b, W_t \leq b) + \mathbb{P}(M_t \geq b, W_t \geq b)$

Hint: you can assume that $\mathbb{E}[\int_0^{\tau_x} h(X_t)dW_t|X_0 = x] = 0$ *for any function* h*. This only holds if* $\mathbb{E}[\tau_x|X_0 = x] < +\infty$*, which is not hard to prove here (you are not asked to do this but will get bonus points of you do).*

We just showed that if there exists a solution $u \in C^2(D)$ to the ODE, then it is necessarily given by (1). Existence is given by the theory of ODEs or PDEs (under some technical assumptions on α, β, f, g) and is out of the scope of the class.

(c) Application 1: Let $dX_t = dW_t$ and define

$$p_x = \mathbb{P}(X_{\tau_x} = b)$$

Show that

$$p_x = \frac{x-a}{b-a}$$

Hint: $p_x = \mathbb{E}[\mathbb{1}_{X_\tau = b}|X_0 = x]$

(d) Application 2: Let $dX_t = dW_t$ and define

$$\bar{t}_{[a,b]}(x) = \mathbb{E}[\tau_x|X_0 = x]$$

Show that

$$\bar{t}_{[a,b]}(x) = (b-x)(x-a)$$