Continuous Time Finance, Spring 2018
NYU Courant Institute
Summary of lecture 2

1 Week 2: Risk Neutral Pricing

Lecture 2 was a bit dense mathematically speaking, so I will highlight the most important parts here, without insisting too much on the technical details. Results are presented in decreasing order of importance, with a grade from 1 (least important) to 10 (most important)

As always, good intuition comes from the discrete time and space model, for example a binomial tree. In that setting, all the formulas we present are easy to derive.

The cornerstone result of the lecture, and the only really important thing to remember is the following:

Risk Neutral Pricing formula and stock Dynamics (importance: $+\infty$)
Given a stock price with dynamics

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$$

where W_t is a Brownian motion under some historical measure $\mathbb{P}$, some interest rate R_t, there exists a measure $\mathbb{Q}$ such that the price at time t of the option maturing at $T > t$ with payoff V_T is given by:

$$V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T R_s ds} V_T | \mathcal{F}_t\right]$$

The stock's dynamics can be rewritten using another Brownian motion in this 'risk neutral' measure $\mathbb{Q}$, $\tilde{W}_t$;

$$dS_t = R_t S_t dt + \sigma_t S_t d\tilde{W}_t$$

How did we get the risk neutral measure?

Risk Neutral measure (importance: 7)
The risk neutral measure was obtained by ensuring that the discounted stock price was a martingale in that measure;

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^t Rudu} S_t | \mathcal{F}_s\right] = e^{-\int_0^s R_u du} S_s$$

In order to do so, we computed $d(e^{-\int_0^t Rudu} S_t)$by Ito's lemma and realized it was of the type

$$d(e^{-\int_0^t R_u du} S_t) = stuff\ (drift\ dt + dW_t)$$

We used Girsanov's theorem to find a measure $\mathbb{Q}$ such that $d\tilde{W}_t = (drift\ dt + dW_t)$ was a Brownian motion.

So why is the risk neutral pricing formula true?

Justification of the risk Neutral formula (importance: 5)
To Justify that such a formula is true, we proceeded to find a strategy Δ_t such that trading according to this strategy yielded the exact payoff V_T at time T, without ever injecting money in our trading scheme (except at initial time);

More precisely, there exists an initial wealth X_t and a strategy Δ_t such that:

$$dX_t = \Delta_t dS_t + (X_t - \Delta_t S_t) R_t dt, \quad \text{(at each time we hold } \Delta_t \text{ shares of stock and put the rest in cash in a self-financing manner)}$$

$$X_T = V_T, \quad \text{almost surely}$$

This means that the value at time t of our wealth X_t, has to agree with the value of the option at itme t, V_t.
Indeed, if we had $X_t < V_t$, sell the option at price V_t, pocket $V_t - X_t > 0$, and use X_t and the strategy Δ_t until time T.
At time T, you will get X_T from the stragey, and you owe V_T from the sale of the option. Since $X_T = V_T$, you will always get from the strategy the exact amount you owe, hence yielding a sure profit of $V_t - X_t$ in any circumstances.
If $V_t > X_t$, do the reverse strategy.

It turns out that in that risk Neutral measure, $D_t X_t$ is a martingale, and hence so is $D_t V_t$ which yields the risk neutral pricing formula.

So how did we get such a strategy Δ_t?

Finding the strategy (importance: 1)
We used the Martingale representation theorem on a judicicously constructed quantity

$$\mathbb{E}_{\mathbb{Q}}[D_T V_T | \mathcal{F}_t]$$