Continuous Time Finance, Spring 2018
NYU Courant Institute
Plan of lecture 1

Preliminaries

I will assume familiarity with the following concepts:
$\sigma-$Algebras, probability measures, probability spaces, random variables, Martingales, convergence of random variables (in L^p, almost surely, in probability, and in law/distribution), Law of Large Numbers, Centrale Limit Theorem, charactersitic functions, Brownian motion and its properties, stochastic integrals w.r.t. Brownian motion (and a martingale more generaly), diffusions, Ito's lemma, SDEs.

1 Week 1: Review of stochastic calculus

1.1 Brownian Motion

- Definition
- Brownian paths (sketch)
- Filtrations w.r.t Brownian motion (or random drivers)
- Martingale property
- Quadratic variation
- Markov property
- Stopping times, running maximum
- Reflection principle

1.2 SDEs

- Mention Stochastic Integrals
- SDEs, paths (sketch), link with ODEs
- Ito's lemma

1.3 Link Between PDEs and SDEs

Let X_t be an Ito diffusion satisfying the SDE;

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

- Feynman Kac
  For some deterministic functions f (running payoff, e.g. dividends) and g (final time payoff), define the average total payoff from the starting date t to maturity T given that the stock started at x;

$$u(t,x) = \mathbb{E}\left[\int_t^T f(s, X_s)ds + g(X_T)|X_t = x\right]$$

Feynman-Kac's theorem states that u is the solution of the following PDE:

$$\begin{aligned} &u_t + a u_x + \frac{b^2}{2} u_{xx} + f = 0, \quad x \in \mathbb{R}, t < T \\ &u(T,x) = g(x) \end{aligned}$$

just for your information, we have more generally for a multidimension diffusion (we won't use this):

$$\begin{aligned} &u_t + a \cdot \nabla u + \frac{1}{2} Trace(bb^T \nabla^2 u) + f = 0, \quad x \in \mathbb{R}, t < T \\ &u(T,x) = g(x) \end{aligned}$$

Proof: Assume u solves the above PDE, compute $du(s, X_s)$ by Ito, integrate from t to T and take $\mathbb{E}[\cdot | X_t = x]$ to get that u is necessarily the function defined above. Existence and uniqueness of a solution for such parabolic PDEs (with some mild assumptions on a, b) allows to concludes rigorously (out of the class' scope).

More generally, we can also include a discount factor r (that can be constant, or a deterministic function of the diffusion), the following function

$$u(t,x) = \mathbb{E}\left[\int_t^T e^{-\int_t^s r(u,X_u)du} f(s, X_s) ds + e^{-\int_t^T r(u,X_u)du} g(X_T) \Big| X_t = x \right]$$

is the solution of the PDE:

$$\begin{aligned} &u_t + a u_x + \frac{b^2}{2} u_{xx} + f - ru = 0, \quad x \in \mathbb{R}, t < T \\ &u(T,x) = g(x) \end{aligned}$$

Proof: Similar to the one above, except apply Ito for $d(e^{-\int_t^s r(u,X_u)du} u(s, X_s))$.

- Boundary value problems
  Let $D \subset \mathbb{R}$ be a bounded domain of $\mathbb{R}$. For some $x \in D$, and given that we start at $X_t = x$ at time t, define the stopping time

  $$\tau_D = \inf\{s > t | X_s \notin D\}$$

  to be the first time we exit from the domain D.
  Define the stopping time

  $$\tau = \min(\tau_D, T)$$

  One can be interested in a quantity similar to the previous one, that is as long as we are in D we get some running payoff f for every unit of time. If we are still in D at time T, then we get some final payoff $g(T, y)$ depending on our final position $y \in D$. If we exited the domain D at τ before the final time, we stop getting the running payoff f and we get a reward $g(\tau, y)$ (potentially 0) depending on where we exited the domain $y \in \partial D$ (boundary of D).

  Please note that g refers to different functions here; $g(T, y)$ is the final payoff, and $g(\tau, y)$ for $y \in \partial D$ is some boundary payoff.
  As an example, let's assume that D is the interval $[a, b]$. So τ is either the first time we exit the interval if we exit before T, or T.
  Let's assume that if we are still in the interval at time T, we for example get a reward

  $$g(T, x) = (x - K)_+$$

  if we happen to hit a before time T, we get a reward

  $$g(\tau, a) = 0$$

and if we exited at b at time τ before T, we receive:

$$g(\tau, b) = T - \tau$$

The above example illustrates that the notation 'g' refers to different functions, depending on its arguments.

Mathematically, our expected reward in the general case is:

$$u(t,x) = \mathbb{E}\left[\int_t^\tau e^{-\int_t^s r(u,X_u)du} f(s,X_s)ds + e^{-\int_t^\tau r(u,X_u)du} g(\tau, X_\tau)\middle| X_t = x\right]$$

Then Feynman-Kac's theorem still apply with the same proof to show that u is the solution of the PDE with boundary conditions:

$$\begin{aligned}
&u_t + au_x + \frac{b^2}{2}u_{xx} + f - ru = 0, \quad x \in D, t < T \\
&u(T,x) = g(x), \quad x \in D \\
&u(\tau,x) = g(\tau,x), \quad x \in \partial D, t \le \tau < T
\end{aligned}$$

- Time independent Boundary value problem
  Assume that we are in the specific case where a, b, f, g, r do not explicitely depend on time;

  $$dX_t = a(X_t)dt + b(X_t)dW_t$$

  and given that we start at $x \in D$ at time 0 (the starting time doesn't matter now since the dynamics of X_t are independentent of it), define the first time we exit the domain

  $$\tau_D = \inf\{t > 0 | X_t \notin D\}$$

  Note that here we are not concerned about some final time payoff (previous situation with $T \to +\infty$). Assume that $\mathbb{E}[\tau_D] < +\infty$.

  Define

  $$u(x) = \mathbb{E}\left[\int_0^{\tau_D} e^{-\int_0^s r(X_u)du} f(X_s)ds + e^{-\int_0^{\tau_D} r(X_u)du} g(X_\tau)\middle| X_0 = x\right]$$

  Then u is the solution of the following PDE (ODE in one dimension):

  $$\begin{aligned}
  &au_x + \frac{b^2}{2}u_{xx} + f - ru = 0, \quad x \in D \\
  &u(x) = g(x), \quad x \in \partial D
  \end{aligned}$$

  Example:
  Assume $X_t = W_t$ is a Brownian motion, and choose $m < M \in \mathbb{R}$. Assume that the brownian motion starts at some $x \in [m, M]$. How long does it takes on average to exit the interval?

  We want the quantity:

  $$u(x) = \mathbb{E}[\tau_D | X_0 = x]$$

  which is an application of the above problem for $a = 0, b = 1, f = 1, g(m) = 0, g(M) = 0, r = 0$. According to our theorem, u solves the ODE:

  $$\begin{aligned}
  &\frac{1}{2}u'' + 1 = 0 \\
  &u(m) = 0 \\
  &u(M) = 0
  \end{aligned}$$

  This is easy to solve and gives $u(x) = -x^2 + c_1 x + c_2$, and we can easily find c_1, c_2 with the conditions $u(m) = u(M) = 0$ to get that $u(x) = (M - x)(x - m)$.

1.4 Monte-Carlo Simulations of SDEs

1.4.1 Euler Scheme

To simulate

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

with a starting point $X_t = x$, set some timestep Δt and denote $t_i = t + i\Delta t$.
Given $X_i = X_{t_i}$ for some i, obtain X_{i+1} by the following first order Euler Scheme:

$$X_{i+1} = X_i + a(t_i, X_i)\Delta t + b(t_i, X_i)\sqrt{\Delta t}Z$$

where Z is a standard normal random variable.

In one dimension, $Z \sim \mathcal{N}(0, 1)$. In multiple dimensions, $Z \in R^n$ is a vector of normal random variables of mean 0, each of them has variance 1, but since the Brownian motions might have some instantaneous correlation ρ_{ij}, so does Z_i and Z_j.

1.4.2 Correlated Gaussians

How do we simulate correlated normal random variables?
Given their covariance matrix Σ, one idea would be to simulate indepndent random variables $Y \in \mathbb{R}^n$ and do a linear combination of its components to get the right correlations for Z; we are seeking for an $n \times n$ matrix α such that

$$Z = \alpha Y$$
$$\mathbb{E}[ZZ^T] = \Sigma$$

Solving this simple system gives the necessary condition

$$\alpha\alpha^T = \Sigma$$

Hence any matrix α satisfying the above property would work. One way of generating a suitable candidate efficiently numerically speaking is to use the Cholesky decomposition of Σ; there exists a triangular matrix L such that

$$\Sigma = LL^T$$

Summary: To simulated correlated normal random variables Z with covariance matrix Σ, first simulate independent random variables Y, and multiply them by the matrix in the Cholesky decomposition of Σ (you can easily get it using some standard numerical linear algebra packages).