ARBITRAGE PRICING
WITH STOCHASTIC VOLATILITY

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Abstract:

We address the problem of pricing contingent claims in the presence of stochastic volatility. Former works claim that, as volatility itself is not a traded asset, no riskless hedge can be established, so equilibrium arguments have to be invoked and risk premia specified.

We show that if instead of trying to find the prices of standard options we take these prices as exogenous, we can derive arbitrage prices of more complicated claims indexed on the Spot (and possibly on the volatility itself).

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1. Introduction

The celebrated Black-Scholes formula is the universal benchmark for option pricing. It relies on a major assumption that the volatility of the Spot is constant (Black & Scholes [1]), or at most a known function of time (Merton [15]). Paradoxically, this assumption is permanently violated, which calls for a theory of option pricing where the volatility itself is non-deterministic.

Work on stochastic volatility has flourished in the last few years. A series of papers in 1987 (Hull & White [12], Scott [17], Johnson & Shanno [13], Wiggins [18]) brought into light the mitigated result that option prices could be computed[1], but at the expense of a stringent assumption: defining a risk premium for the volatility. It tells us what prices would be in an economy where agents have some types of preferences, but offers no way to lock these prices, i.e. to make sure profits if a price deviated from the theory.

Our goal is to expose in a simple fashion the ingredients allowing for arbitrage pricing with stochastic volatility, with no need for any volatility risk premium to be specified.

The current situation is in many ways similar to that of the interest rate theory before its recent developments (Ho & Lee [11], Heath, Jarrow & Morton [10], El Karoui, Myneni & Viswanathan [7]), which mainly consisted of a shift from equilibrium considerations to arbitrage arguments. The aim is not to explain the yield curve, but, taking it for granted and along with evolution assumptions, to obtain indisputable arbitrage prices for derivative securities.

Our approach straightforwardly mimics this path in that we do not try to explain standard option prices observed in the market: we merely use the fact that they are traded assets, offering (once again along with stochastic assumptions) the basis for shaping higher order assets, such as forward start options or any path-dependent claim.

More precisely, the problem we address here is: finding the fair price for claims contingent on both the Spot price and its volatility (or only one of them), hedging them, i.e. exhibiting a trading strategy that ensures a perfect replication.

The outline of this paper is as follows: In Section 2, we obtain the prices of logarithmic profiles from the standard Call prices. In Section 3, we introduce the forward variance markets and compute their arbitrage free values from the prices of Section 2. Section 4 introduces stochastic assumptions on the forward variance process, and establishes the risk neutral volatility and Spot processes. Section 5 presents option pricing and hedging,

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[1] Moreover, they can be expressed as weighted average of deterministic volatility option prices if the volatility and the Spot are independent.

which involves numerical techniques addressed in Section 6. Section 7 explores some extensions, including stochastic correlation and Section 8 concludes.

2. Prices of logarithmic profiles from Call prices`

For the sake of simplicity, the interest rate is assumed to be zero at all times. This hypothesis greatly eases the presentation, and can be relaxed with no serious damage to the theory (see Section 7). We thus consider a frictionless market with a riskless asset B (bond) bearing no interest, and a risky security S, the price of which at time t is denoted S_t (Spot process).

We associate to a real valued function f and to a time T the financial instrument f_T delivering $f(S_T)$ at time T. Its value at time $t < T$ is denoted $f_T(t)$. A Call option of maturity T and exercise price (Strike) K is a claim that promises $C_K(S_T) = \text{Max}(S_T - K, 0)$ at time T. It will hereafter be denoted $C_{K,T}$.

We assume that the continuum of all $(C_{K,T})_{K,T}$ are traded[2] and that their prices at time 0 (i.e. today) $(C_{K,T}(0))_{K,T}$ are consistent with no arbitrage[3].

For all T, the knowledge of $(C_{K,T}(0))_K$ allows us to obtain ϕ_T, the risk neutral distribution of S_T, which in turn determines the price of any European contingent claim (defined through the end of period payoff). The first step is accomplished by considering a simple portfolio of Calls (known as "butterfly" in financial markets):

$$\frac{C_{K-\varepsilon,T} - 2C_{K,T} + C_{K+\varepsilon,T}}{\varepsilon^2}$$

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[2] This is not so far from reality, on O.T.C. Foreign Exchange markets, for example. Market makers are supposed to give two-way prices for all strikes and maturities.

[3] This hypothesis can be proved to be equivalent to the following conditions:

- For all T, $C_{K,T}(0)$ is convex decreasing towards 0 in K,

  At $K = 0$, $C_{K,T}(0) = S_0$ and $\frac{\partial C_{K,T}}{\partial K}(0) \geq -1$

- For all K, $C_{K,0}(0) = \text{Max}\ (S_0 - K, 0)$

- For all $T_1 < T_2, C_{K,T_1}(0) \leq C_{K,T_2}(0)$

More generally, this latter condition must hold for any convex function of S.

As ε tends towards zero, its profile converges to a Dirac function at point K (Arrow-Debreu security associated with state K), and its price converges to $\phi_T(K)$ if finite (if not, there is a lump distribution on K). We thus get a simple description of ϕ_T from $(C_{K,T})_K$:

$$\phi_T(K) = \frac{\partial^2 C_{K,T}}{\partial K^2}$$

The equality is to be taken in a distributional sense (C is a convex function of K, hence admits left and right first derivatives). We refer to Breeden & Litzenberger [3] for more details.

Denoting P_T the associated probability (which density function is ϕ_T), we can compute the price at time 0 of any profile f in $L^1(P_T)$ through:

$$f_T(0) = \int_0^\infty f(S_T)\phi_T(S_T)\,dS_T \equiv E^{P_T}[f]$$

In particular, if L_T, a claim delivering the logarithm of S_T at time T, belongs to $L^1(P_T)$,

$$L_T(0) = E^{P_T}[\ln S_T]$$

We thus obtain, under appropriate integrability conditions, the existence of $(L_T(0))_T$, which is actually the only piece of information we need for the sequel. The family $(C_{K,T}(0))_{K,T}$ was introduced because of its daily use by the practitioners, but only served our purpose as a device in view of obtaining $(L_T(0))_T$, which will account of the term structure of volatility.

A word of caution is in order: the knowledge of $(C_{K,T})_{K,T}$ is <u>equivalent</u> to the knowledge of $(\phi_T)_T$. However, we can go no further, i.e. deducing the risk neutral <u>diffusion process</u>[4], as two different processes may generate the same distributions at all times. Therefore, given only $(C_{K,T})_T$, we cannot price path-dependent or American options, nor can we compute the dynamic hedging parameters. Without any additional assumption, we can merely price by arbitrage (possibly infinite) combinations of Calls.

3. Arbitrage free value of forward variance

We now add structure to the picture by classically introducing a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where $(\mathcal{F}_t)_t$ is a right continuous filtration containing all P-null sets, $\mathcal{F}_0$ being

[4] However, recent work shows that if we restrict to one dimensional Ito diffusions with a specified risk-neutral drift, a unique diffusion process is obtained (Dupire [6]).

trivial, and a Spot price process S_t governed by the following stochastic differential equation:

(3.1) $$\frac{dS}{S} = \mu_t \, dt + \sigma_t \, dW_{1,t},$$

where W_1 is a P-Brownian motion adapted to $(\mathcal{F}_t)_t$, and μ_t and σ_t may in turn be stochastic processes measurable and adapted to $\mathcal{F}_t$[5].

By Ito's lemma,

$$d \ln S_t = \frac{dS_t}{S_t} - \frac{\sigma_t^2}{2} dt,$$

which, integrated between T_1 and T_2, yields

$$\ln S_{T_2} - \ln S_{T_1} = \int_{T_1}^{T_2} \frac{dS_t}{S_t} - \frac{1}{2} \int_{T_1}^{T_2} \sigma_t^2 \, dt,$$

which we rewrite as

(3.2) $$\int_{T_1}^{T_2} \sigma_t^2 \, dt = 2 \int_{T_1}^{T_2} \frac{dS_t}{S_t} - 2(\ln S_{T_2} - \ln S_{T_1}).$$

The stochastic integral $\int_{T_1}^{T_2} \frac{dS_t}{S_t}$ can be interpreted as the wealth at time T_2 of a strategy consisting of permanently keeping one unit of the riskless bond B invested in the risky asset S between time T_1 and time T_2 (through the possession of $\frac{1}{S_t}$ units of S). This strategy is self financing, for the interest rate is zero.

The left hand side of (3.2) is the cumulative instantaneous variance of the Spot return between T_1 and T_2. It can be reproduced by the portfolio $2(L_{T_2} - L_{T_1})$, associated with a dynamic self-financing strategy consisting in keeping 2 units of the riskless asset B permanently invested in the risky asset S between T_1 and T_2.

If a contract delivering $\int_{T_1}^{T_2} \sigma_t^2 \, dt$ at T_2 is traded at time t, it therefore has a unique possible value, equal to $2(L_{T_1}(t) - L_{T_2}(t))$, which is determined by arbitrage. In other words, if its market value were different, definite profits could then be generated. This means that even if there is no such forward market, we can nevertheless synthesize it; we will therefore assume it exists.

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[5] For the sake of simplicity, integrability requirements will not be stressed and we assume a terminal time for the economy. We refer to Karatzas & Shreve [14] for a clear presentation of stochastic calculus.

Let V_T be a forward contract on the instantaneous variance to be observed at time T. From the above relation, we get by differentiation the value of the contract V_T at any time $t < T$:

$$V_T(t) = -2\,\frac{\partial L_T(t)}{\partial T}$$

As $-L_T$ is a convex function of S, $V_T(t)$ is indeed positive (see footnote 3). From the values $(L_T(0))_T$ of Section 2, we are now able to deduce the initial instantaneous forward variance curve:

$$V_T(0) = -2\,\frac{\partial L_T(0)}{\partial T}$$

4. Stochastic assumptions on the forward variance and risk neutral processes

At this point, more stochastic assumptions are needed. While it may seem natural to apply them on the instantaneous volatility, this unfortunately leads to the undesirable necessity of specifying risk premia.

It is simpler, along the lines of Heath, Jarrow & Morton [10], to directly model the forward variance, which automatically ensures compatibility with $(L_T(0))_T$. We make, among many other possible choices, the assumption that $V_T(t)$ is Lognormal. Thus:

$$\frac{dV_T(t)}{V_T(t)} = a\,dt + b\,dW_{2,t} \tag{4.1}$$

where a and b are constant[6] and W_2 is another Brownian motion adapted to $(F_t)_t$, possibly correlated with W_1. We now look for the risk-neutral process of several quantities. By risk-neutral process, we understand expressing the dynamics with a Brownian Motion that can be "traded" in a self financing way.

4.1. Risk neutral process for the forward variance

Defining $dW'_{2,t} \equiv dW_{2,t} + \frac{a}{b}\,dt$, the equation (4.1) can be rewritten as

$$\frac{dV_T(t)}{V_T(t)} = b\,dW'_{2,t} \tag{4.2}$$

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[6] In fact, b can be a deterministic function of time, which would allow for mean reversion on the *instantaneous* volatility (see Section 7.4).

where $W_2^{'}$ is a Brownian motion under Q_2, the P-equivalent probability classically obtained by Girsanov's Theorem.

Applying Ito's lemma to (4.2):

$$d\ln V_T(t) = -\frac{b^2}{2}\,dt + b\;dW_{2,t}^{'}$$

Integrating between 0 and t leads to

$$\ln V_T(t) = \ln V_T(0) - \frac{b^2}{2}\,t + b\;W_{2,t}^{'} \tag{4.3}$$

Indeed, under Q_2, V_T is a martingale:

$$E^{Q_2}\left[V_T(t)|\,V_T(0)\right] = V_T(0)$$

4.2. Risk neutral processes for the instantaneous variance and volatility

Let us define $v_t \equiv V_t(t) = \sigma_t^2$, the instantaneous variance at time t. From (4.3), we obtain

$$\ln v_t = \ln V_t(0) - \frac{b^2}{2}\,t + bW_{2,t}^{'}$$

which can be differentiated into

$$d\ln v_t = \left(\frac{\partial \ln V_t(0)}{\partial t} - \frac{b^2}{2}\right)dt + b\,dW_{2,t}^{'} \tag{4.4}$$

and finally gives, by Ito's lemma

$$\frac{dv_t}{v_t} = \frac{\partial \ln V_t(0)}{\partial t}\,dt + b\,dW_{2,t}^{'} \tag{4.5}$$

This last expression is the risk neutral process of the instantaneous variance (the real process is not needed), and provides a way to estimate the parameter b from the single Spot process. The drift term $\frac{\partial \ln V_t(0)}{\partial t}$ ensures compatibility with the initial volatility term structure given by $(L_T)_T$.

We can also derive the risk neutral process for the instantaneous volatility, which is Lognormal as the instantaneous and forward variances. From relation (4.4),

$$d\ln \sigma_t = \frac{1}{2}\left(\frac{\partial \ln V_t(0)}{\partial t} - \frac{b^2}{2}\right)dt + \frac{b}{2}dW_{2,t}^{'}$$

and thanks to Ito's lemma once again,

(4.6) $$\frac{d\sigma_t}{\sigma_t} = \left(\frac{1}{2} \frac{\partial \ln V_t(0)}{\partial t} - \frac{b^2}{8} \right) dt + \frac{b}{2} dW'_{2,t}$$

At this point, we can price any claim contingent on the volatility. The hedge will be accomplished through a dynamic trading of $(L_T)_T$. This allows us to "trade" W'_2.

4.3. Risk neutral process for the Spot

The Spot process follows the following stochastic equation (3.1):

$$\frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dW_{1,t}$$

The interest rate being zero, the risk neutral process for the Spot, which actually governs prices, is (as it will be made clear in Section 5):

(4.7) $$\frac{dS_t}{S_t} = \sigma_t \, dW'_{1,t}$$

with σ_t satisfying (4.6) and

$$dW'_{1,t} \equiv dW_{1,t} + \frac{\mu_t}{\sigma_t} dt$$

At this stage, a most important point has to be stressed: to obtain the risk neutral process for the Spot. Cancelling its drift is not enough, as the instantaneous volatility also has to be replaced by the risk neutral one established earlier in this section.

5. Option pricing and hedging under stochastic volatility

5.1.Pricing

We now make the additional assumption that $(\mathcal{F}_t)_t$ is the natural augmented filtration associated with W_1 and W_2. The filtration associated with W'_1 and W'_2 conveys the same information and therefore is the same. We define Q as the P-equivalent measure, defined through Girsanov's Theorem[7], under which $W_1^{'}$ and $W_2^{'}$ are Q-Brownian motions. The processes S_t and V_T are Q-martingales: we recall from Section 4 that

$$\frac{dS_t}{S_t} = \sigma_t \, dW_{1,t}^{'}$$

with

$$\frac{d\sigma_t}{\sigma_t} = \left(\frac{1}{2} \frac{\partial \ln V_t(0)}{\partial t} - \frac{b^2}{8} \right) dt + \frac{b}{2} dW_{2,t}^{'}$$

and

$$\frac{dV_T(t)}{V_T(t)} = b \, dW_{2,t}^{'}$$

We can then express $dW_1^{'}$ and $dW_2^{'}$ in terms of dS and dV_T:

$$dW_{1,t}^{'} = \frac{dS_t}{\sigma_t S_t} \tag{5.1}$$

$$dW_{2,t}^{'} = \frac{dV_T(t)}{bV_T(t)} \tag{5.2}$$

Let A be a contingent claim that delivers in T a payoff dependent on the paths followed by the Spot and variance, i.e. a square integrable F_T-measurable random variable on Ω. We then define h by:

$$h(t) \equiv E^Q[A | F_t]$$

Clearly, h is a Q-martingale adapted to $(F_t)_t$. It can therefore be represented (Karatzas & Shreve [14]) by:

$$h(t) = h(0) + \int_0^t \alpha_u \, dW_{1,u}^{'} + \int_0^t \beta_u \, dW_{2,u}^{'}$$

The stochastic integrals can be expressed in terms of dS and dV_T thanks to (5.1) and (5.2):

$$h(t) = h(0) + \int_0^t \frac{\alpha_u}{\sigma_u S_u} dS_u + \int_0^t \frac{\beta_u}{bV_T(u)} dV_T(u)$$

They can therefore be interpreted as a self financing trading strategy on the Spot and V_T. The initial wealth $h(0)$, associated with this dynamic trading strategy, gives exactly the contingent claim A at time T.

Therefore:

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[7] If W_1 and W_2 are correlated, some care is needed, for instance performing the transformation on orthogonalized Brownian motions. However, $W_1^{'}$ and $W_2^{'}$ will exhibit the same correlation as W_1 and W_2.

1. The price of A at time 0 is $h(0) = E^Q[A \mid F_0]$.

2. There exists a dynamic hedging strategy.

In the terminology of Harrison & Kreps [8], there exists a self-financing portfolio which transforms $h(0)$ into A at time T. The value of the claim at time t is given by

(5.3) $$h(t) = E^Q[A \mid F_t].$$

where the expectation is taken over all paths for S and σ, generated by the risk neutral processes of Section 4.

From the knowledge of $(L_T(0))_T$, we derived the values of $(V_T(0))_T$. Given some stochastic assumptions on V_T, we were able to compute the value of any path-dependent option, and, in particular, values for standard European Calls.

However, if $(L_T(0))_T$ were obtained from the market Call prices, one should not expect to recover the latter exactly through this computation[8], as the 2-dimensional information $(C_{K,T}(0))_{K,T}$ was compacted into a 1-dimensional temporal one, $(L_T(0))_T$, in the process[9]. The transversal information $(C_{K,T}(0))_K$, i.e. the deviation from the Black-Scholes prices, may provide a way complementary to (4.5) to estimate b, the volatility of the variance.

5.2.Hedging

As we deal with a one factor model, all forward variances are perfectly correlated and theoretically any one of them could be used to hedge volatility risk. Moreover, any option would do though it entails an undesired associated Spot position that must be filtered out. This one factor assumption is blatantly unrealistic for practitioners daily observe a wide variety of twists in the volatility curve, as given by the L_T. It is therefore important to elaborate a maturity by maturity hedge.

This can be performed by locally altering the volatility curve and computing the associated option price change. We thus obtain a theoretical equivalent position in L_T which, when expressed in term of $(C_{K,T})_K$, would lead to a portfolio comprising an infinite number of calls. Indeed in practice, only a few of them may be used, even though leading to a high hedging cost due to imperfect liquidity. However from a global portfolio risk-management viewpoint, this maturity analysis of partially offsetting positions allows

[8] We actually get "Hull & White" prices for European Calls.

[9] This should not be a point of worry, for it parallels the fact that the yield curve is computed from Bond prices that cannot be exactly recovered from that curve. However, it is possible to retrieve the exact price even with a deterministic volatility model! (see Dupire [6])

to precisely decompose the volatility risk through the timescale and to decide whether action should be taken or not.

6. Numerical methods

6.1.Monte Carlo

In the general case, (5.3) has to be evaluated numerically, by Monte-Carlo simulations (Boyle [2]), which, in its simplest form, runs as follows:

(a) generation of a path for σ (through a path of $W_2^{'}$),

(b) generation of a path for S (through a path of $W_1^{'}$), depending on (a) and the correlation between the two Brownian Motions.

(c) computation of the terminal payoff for these sample paths,

(d) iterating thousands of times the first three steps and averaging the values obtained from (c).

Indeed, if A merely depends on σ, step (b) can be omitted. Hedging parameters can be computed at the same time through a small shift of the same paths.

6.2.Discretisation

The risk-neutral process for S and σ can be used to write the two dimensional (in S and σ) partial differential equation and discretised it implicitly, which is somewhat cumbersome, even with ADI or Hopscotch methods.

We prefer to expose an explicit discretisation of the diffusion process followed by S and σ. Two main features are required:

(1) the discrete scheme must exhibit the same mean and covariance matrix as the continuous model.

(2) The scheme must be recombining, preventing an intractable exponential explosion.

To discretise a process, the most straightforward approach is to generate from one node (mother node) other nodes (daughter nodes) as to verify condition (1). For instance, both the logarithm of the Spot and the volatility can be discretised binomially (mean plus or minus one standard deviation), leading to a rectangular scheme with a pattern of probability reflecting the correlation ($\frac{1+\rho}{4}$ on the first diagonal, $\frac{1-\rho}{4}$ on the second one).

Unfortunately, if on two following steps the volatility increases then decreases, the path corresponding to an up move of the Spot followed by a down move will not recombine with the path obtained by the reverse order. On the second step, the volatility is higher and the two paths will therefore cross, violating condition (2).

There are several ways to overcome this difficulty. Probably the simplest is the following:

We discretise the instantaneous volatility binomially but the Spot trinomially. A trinomial scheme has the great advantage of being able to cater for a wide range of variance. This means that we can define a rectangular grid at each time step, with the Spot discretisation step calibrated on the average volatility level. For extreme values of volatility, the one time step variance of the continuous time process cannot fit into the discrete scheme but this can be safely ignored by crystallising the volatility above the threshold, for probabilities in these portions of the grid are very low.

Once the grid is built and the connections from a mother to its six daughters are defined, we proceed as usual to value European or American claims. We first compute its value at the last time step, then proceed backwards in time to obtain at each node the (discounted) expected value (bounded by the intrinsic value, in the case of American options), until we reach the root node where the desired premium is obtained.

6.3.Analytical approximation

In the case of no correlation, we can consider that the volatility trajectory is drawn first and get the value of the claim conditioned on that trajectory now seen as deterministic. For European Calls, the conditioned price merely depends on the cumulative variance (sum of the instantaneous variance throughout the life of the option), so we can express the full price as an expectation of deterministic prices with different variance parameters. We then get the price as a one dimensional integral:

$$C_0 = \int_0^\infty C_{BS}(V)\psi(V)\,dV$$

where ψ is the density of V, the cumulative variance and C_{BS} is the Black-Scholes price with the associated constant volatility. This has been clearly exposed by Hull&White [12] who proposed an analytic approximation based on the Taylor expansion up to the second order of the Call price as a function of the cumulative variance. Actually, the same accuracy can be achieved by instead approximating the density by a sum of two Dirac masses of .5 located one standard deviation away on each part of the mean. This means that we obtain a quite accurate approximation by taking the average of two Black-Scholes prices, which practitioners easily adopt.

This representation of the stochastic volatility price as a weighted average of deterministic volatility prices applies indeed to any European claim (i.e. pay-off contingent on the final value of the spot) but also extends to slightly more complicated instruments, for instance forward start options which appear in the popular "cliquet" or ratchet structure. A forward start option grants at time T_2 the amount $\mathrm{Max}(S_{T_2} - S_{T_1}, 0)$. Its stochastic volatility price is the average over values of the cumulative variance between T_1 and T_2 of the deterministic prices and can be as well approximated by an average of two deterministic prices.

However some care should be taken in the path dependent case in general. For instance, it is tempting but wrong to write the stochastic price of a barrier option as an average of the deterministic prices in the case of non-zero interest rates. In effect, two volatility trajectories that exhibit the same cumulative variance will not necessarily yield the same option value, for the associated time change may destroy certain spot trajectories.

7.Extensions

7.1.Non-zero interest rates

Section 3 showed that the forward variance could be synthesized through

- the initial purchase of a portfolio
- a dynamic strategy of holding the asset

In the case of stochastic interest rates, this strategy will consist in investing at time t $B(t, T_2)$, value at time t of a zero-coupon bond of maturity T_2, in the risky asset. The strategy is not self financing any more but its cost can be exactly assessed independently of the model.

Unfortunately, the interests at time T_2 of the proceeds of the portfolio at time T_1, depend both on the value at time T_1 of the spot and of the interest rates. We then need a model of interest rate, together with the correlation between spot and rates. Once the risk neutral processes for the spot, the interest rates and the volatility are obtained, we make use of the martingale representation of Section 5, now with three Brownian motions, to deduce contingent claim prices together with the replicating hedge.

In the case of deterministic interest rates, computations run smoothly with no major changes with respect to the zero rate case.

7.2.Stochastic correlation

Multifactor models now proliferate due to two reasons. Firstly, more and more abstruse cross-market instruments are proposed and dealt by investment banks, notedly "quanto" structures which pay in one currency an intrinsic value expressed in another, "best of" and spread options. Secondly, to finely tune the pricing and hedging of standard options, it is important to infuse some stochasticity on parameters otherwise assumed to be constant or deterministic functions of time.

When a modelling entails several Brownian Motions, correlations indeed come into play and may themselves be stochastic. We consider the case of two underlying assets *X* and *Y*, traded on the market as well as European Call options written on them. The analysis of the preceding sections tells us we can price and hedge path dependent or American options on each of the underlying assets in an arbitrage free fashion. We now pay attention to contingent claims written on <u>both</u> assets, in which case prices are affected by the correlation as well. We address the problem of pricing and hedging under stochastic correlation. It should be emphasised that this correlation is the one between the two Brownian Motions associated with *X* and *Y* and not the one between the Brownian Motions of the Spot and of the variance.

Practitioners daily manipulate volatility and increasingly correlation. However, from a mathematical stand-point, the more natural notions are variance and covariance: whenever a correlation enters a formula, it is with the product of the associated volatilities. It is unfortunately somewhat constraining to make on the covariance a stochastic assumption which is compatible with the two volatilities (inferior to their product). We therefore prefer to make an assumption on the stochastic evolution of correlation, or more precisely of the forward correlation. To keep the correlation between -1 and 1, we express it as the cosine of a stochastic angle. We assume that European Calls are traded on both *X* and *Y*, which gives us the forward variances V_T^X and V_T^Y as in Section 3. Moreover, we assume that cross-markets instruments as spread options or quantos are traded, which gives us the forward covariance V_T^{XY}. We <u>define</u> RO_T^{XY}, the forward correlation as the forward covariance divided by the squared root of the product of the two forward variances:

$$RO_T^{XY} \equiv \frac{V_T^{XY}}{\sqrt{V_T^X V_T^Y}}$$

It can be interpreted as the instantaneous forward correlation but, though computed from arbitrage values, it is not in itself an arbitrage value! We make, along with stochastic assumptions on V_T^X and V_T^Y homologous to (4.1), stochastic assumptions on R_T^{XY}, or more precisely on the angle of which it is the cosine. We hence deduce the stochastic differential equation for V_T^{XY} by applying Ito's lemma to the relationship:

$$V_T^{XY} = RO_T^{XY}\sqrt{V_T^X V_T^Y}$$

As V_T^X, V_T^Y and V_T^{XY} are martingales under the risk neutral probability, we obtain their risk neutral dynamics by simply cancelling the drifts. The next step consists in deriving the risk neutral dynamics of the instantaneous (not forward) quantities. We saw how to do it for the variance in Section 4. For the covariance, an analogous computation leads from:

$$\frac{dV_T^{XY}}{V_T^{XY}} = c\ dW''$$

to

$$\frac{dv_T^{XY}}{v_T^{XY}} = \frac{\partial \ln V_T^{XY}(0)}{\partial T}dt + c\,dW''$$

and we are now in a position to derive the risk neutral process of the instantaneous correlation by another application of Ito's lemma to the relation:

$$\rho^{XY} = \frac{v^{XY}}{\sqrt{v^X v^Y}}$$

The dynamics of the joint process (X,Y) is then fully defined and we can make use of the martingale representation of Section 5 with now five Brownian Motions, associated to X, Y, σ_X, σ_Y and ρ^{XY}.

7.3.Diffusion-jumps models

In a precursor work, Merton [16] considered jumps of random amplitude occurring at random Poisson points in time, in addition to a diffusion Black-Scholes model. The model is not complete and demands the assessment of risk premia. Once done, stochastic prices are computed as the expectation of deterministic prices.

Our assumptions do not allow us to make Merton model complete. However, in the simplified case where the amplitude of the jump is deterministic, completeness is achieved.

7.4.Mean reverting volatility

We now allow volatility of V_t to be time dependent instead of constant as in Section 4. The risk neutral dynamics of V_t are now:

$$\frac{dV_T(t)}{V_T(t)} = b(t,T)\ dW_{2,t}^{'}$$

Along the lines of Section 4, we get

$$\ln v(t) = \alpha(t) + \int_0^t b(s,t)\ dW_{2,s}^{'}$$

where $$\alpha(t) \equiv \ln V_t(0) - \frac{1}{2}\int_0^t b^2(s,t)\ ds$$

and $$d\ln v(t) = (\alpha'(t) + \int_0^t \frac{\partial b}{\partial t}(s,t)\ dW_{2,s}^{'})\ dt + b(t,t)\ dW_{2,s}^{'}$$

The presence of a stochastic integral in the drift of $d\ln v$ is troublesome, introducing in general path dependency, except in the case where it can be expressed as a function of $\ln v(t)$. As both the integral and $\ln v(t)$ are Gaussian, the relationship has then to be affine and there is necessarily a function of time λ such that:

$$\int_0^t \frac{\partial b}{\partial t}(s,t)\ dW_{2,s}^{'} = -\lambda(t)\int_0^t b(s,t)\ dW_{2,s}^{'}$$

and by equating the terms in $dW_{2,s}^{'}$ and integrating, we get:

$$b(t,T) = b(t,t)e^{-\int_t^T \lambda(s)\,ds}$$

and the stochastic differential equation for ln v can be restated as

$$d\ln v(t) = \left[\alpha'(t) - \lambda(t)(\ln v(t) - \alpha(t))\right] dt + b(t,t)\ dW_{2,t}^{'}$$

In other words, the logarithm of the instantaneous variance follows an Ornstein-Uhlenbeck process with a pull-back force of λ.

8. Conclusion

In this paper, we started by assuming that European Calls of all Strikes and maturities were traded, and that their market prices were consistent with no arbitrage. From these prices, we deduced the arbitrage price L_T of contingent claims that promised ln S_T at date T. These prices L_T were then shown to be related to the instantaneous variance of the Spot return process and furthermore permitted to uniquely set the value of a forward market on these instantaneous variances, at any maturity. In other words, it was shown that such forward markets could be synthesized from the mere knowledge of the $(L_T)_T$.

The use of these synthesized markets was the key that could open the door of volatility hedging and pricing. At this point, we needed to set assumptions on the variance process to go further. We focussed on a one factor model (in which all forward markets and current variance are perfectly correlated) and derived the risk neutral process for both the instantaneous variance and the Spot itself.

We obtained arbitrage free prices that do not depend on any risk premia nor on a volatility drift. They do depend on the term structure of volatility, on the correlation between Spot and volatility, and on the volatility of the latter. This was achieved by martingale methods, thanks to the ability to

- perform the integral representation, through the choice of the appropriate filtration,
- interpret it as a self financing strategy, due to the possibility of "trading" the two Brownian motions involved.

Any claim measurable with respect to the associated σ-field can be dynamically spanned and therefore fairly priced.

In a more general way, to achieve pricing and hedging, we need the risk neutral dynamics of all the stochastic variables. To ensure compatibility with current known parameters, it is easiest to make an assumption on the forward value of the variable and then proceed on with two steps: firstly obtain the risk neutral dynamics of the forward value and then get the risk neutral dynamics of the instantaneous value.

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