PRICING AND HEDGING WITH SMILES

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Black-Scholes volatilities implied from market prices exhibit a strike pattern, commonly termed "Smile", as well as a term structure. This non constancy of volatility contradicts the assumptions of the model and leads to the unpleasant situation where a single spot process has many supposedly constant but yet distinct volatilities.
We show how to reconcile these seemingly incompatible assumptions with a single hypothesis on the spot process (instantaneous volatility which is a deterministic function of spot and time), which has the merit of preserving one-dimensionality and completeness. This process is used to price exotic options and hedge them robustly with standard European options.

1. Introduction

Option pricing consists mainly, after having specified a model and estimated its parameters, of deriving option prices (unique if the market model is complete) as a function of these parameters. A prototypical example is given by the Black-Scholes [1] model which we will use as a guideline. It gives us options prices as a function of a parameter called volatility. We often have to invert this relationship, for what we know is the price of the option, given by the market. We thus get the implied value of the parameter.

If the model were good, this implied value would be the same for all option market prices, a fact that reality crudely denies us. Implied Black-Scholes volatilities strongly depend on the maturity and the strike of the European option under scrutiny. If the implied volatilities of at the money options on the Nikkei are 20% for a maturity of 6 months and 18% for a maturity of 1 year, we are in the uncomfortable situation of assuming at the same time that the Nikkei vibrates with a constant volatility of 20% for six months and that the same Nikkei vibrates with a constant volatility of 18% for one year.

It is easy to solve this paradox by allowing volatility to be time dependent, as Merton [13] did long ago. The Nikkei would firstly exhibit an instantaneous volatility of 20% and subsequently a lower one, computed by a forward relationship to accommodate the one year volatility. We now have one unique process, compatible with the two option prices. From the term structure of implied volatilities we can infer a time dependent instantaneous volatility, for the former is the quadratic mean of the latter. The spot process S is then governed by the following stochastic differential equation:

$$\frac{dS}{S} = r(t)\,dt + \sigma(t)\,dW$$

where r is the instantaneous forward rate implied from the yield curve.
Some Wall Street houses incorporate this temporal information in their discretization schemes in order to price American or path-dependent options.

However, the dependence of implied volatility on the strike, for a given maturity (known as the Smile effect) is trickier. Many researchers have attempted to enrich the Black-Scholes model to compute a theoretical "Smile". Unfortunately they have to introduce a non-traded source of risk (jumps in the case of Merton [14] and stochastic volatility in the case of Hull and White [8]) thus losing the completeness of the model. Completeness is of the highest value; it allows for arbitrage pricing and hedging.

We address the following natural question: Is it possible to build a spot process which:

a) is compatible with the observed Smiles at all maturities?
b) keeps the model complete?

More precisely, given the prices of European Calls of all strikes K and maturities T: $C(K,T)$, is it possible to find a risk neutral process for the spot in the form of a diffusion,

$$\frac{dS}{S} = r(t)\,dt + \sigma(S,t)dW$$

where the instantaneous volatility σ is a deterministic function of the spot and of the time?

This would nicely extend the Black-Scholes model, to take full power of its diffusion setting, without increasing the dimension of the uncertainty. We would have the features of a one factor model (hence easily discretizable) to explain all European option prices. We could then price and hedge any American or path-dependent options[1]. We could thus answer questions like "how to hedge a forward start option?" or "what is the Smile of Asian options?" or "which strike to use to hedge the volatility risk on the intermediate date of a compound option?"

In the second section we review a few basic facts. We address the problem in a continuous time setting in the third section and in discrete time and price space in the fourth section. Hedging issues are tackled in the fifth section and concluding remarks take place in the final section.

2. The problem

If the spot price follows a one dimensional diffusion process, then the model is complete and option prices can be computed by discounting an expectation with respect to a so-called "risk neutral" probability under which the discounted spot has no drift (but retains the same diffusion coefficient).

More precisely, path-dependent options are priced as discounted expected value of their terminal payoff over all possible paths. In the case of European options, it boils down to an expectation over the terminal values of the spot (which can be seen as bundling the paths which end at a same point).

It follows that the knowledge of the prices of all path-dependent options is equivalent to the knowledge of the full (risk neutral) diffusion process of the spot, while knowing all European option prices merely amounts to knowing the laws of the spot at different times, conditional on its current value.

The full diffusion contains much more information than the conditional laws, as distinct diffusions may generate identical conditional laws. However, if we restrict ourselves to risk neutral diffusions, the ambiguity is removed and we can retrieve from the conditional laws the unique risk neutral diffusion they come from. This result is interesting on its own but we will exploit its consequences in terms of hedging as well.

3. A diffusion from prices

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[1] Even for European options, the knowledge of the whole process is compulsory to hedge.

In this section, we address the problem of existence, uniqueness and construction of a diffusion process compatible with observed option prices, in a continuous time setting. To gain considerably in clarity without losing much in generality, we assume that the interest rate is 0.

3.1 From prices to distributions

For a given maturity *T*, the collection $C(K,T)_K$ of option prices of different strikes yields the risk neutral density function φ_T of the spot at time *T* through the relationship:

$$C(K,T) = \int_0^\infty (x-K)^+ \varphi_T(x)\,dx$$

which we differentiate twice with respect to *K* to obtain:

$$\varphi_T(K) = \frac{\partial^2 C}{\partial K^2}(K,T)$$

If we start from (S_0, T_0), we have $\varphi_{T_0}(K) = \delta_{S_0}(K)$ for $C(K,T_0) = (S_0 - K)^+$.

We are then left with an interesting stochastic problem (with the notation (x,t) instead of (K,T)):

Knowing all the densities conditional on an initial fixed (x_0, t_0), is there a unique diffusion process which generates these densities?

The converse problem is well known: from the coefficients *a* and *b* (satisfying slow growth assumption) of the diffusion:

$$dx = a(x,t)\,dt + b(x,t)\,dW$$

we can deduce the conditional distributions φ_t thanks to the Fokker-Planck (or forward Kolmogorov) equation (define $f(x,t) \equiv \varphi_t(x)$):

$$\frac{1}{2}\frac{\partial^2 (b^2 f)}{\partial x^2} - \frac{\partial (af)}{\partial x} = \frac{\partial f}{\partial t}$$

However, a diffusion is more informative than the distributions it generates. It is easy to exhibit two distinct diffusions which generate the same distributions. For instance, with $x_0 = 0, t_0 = 0$:

$$dx = -\lambda x\,dt + \mu\,dW$$

and

$$dx = \mu e^{-\lambda t}\,dW$$

lead to the same Gaussian distribution for each t , with a mean equal to 0, and a variance equal to $\frac{\mu^2}{2\lambda}(1 - e^{-2\lambda t})$.

However, if we restrict ourselves to risk-neutral diffusions, we can recover, up to technical regularity assumptions, a unique diffusion process from the $f(x,t)$. The interest rate being 0, we only pay attention to martingale diffusions (i.e. $a = 0$), which in the case of our counterexample rules out the first candidate.

3.2 From distributions to the diffusion

The Fokker-Planck equation then takes the simple form (now f is known and b is the unknown!):

$$\frac{1}{2}\frac{\partial^2(b^2 f)}{\partial x^2} = \frac{\partial f}{\partial t}$$

As f can be written as $\frac{\partial^2 C}{\partial x^2}$, we obtain, after changing the order of derivatives:

$$\frac{1}{2}\frac{\partial^2(b^2 f)}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial C}{\partial t}\right)$$

Integrating twice in x for a constant t gives:

$$\frac{1}{2}b^2_{\alpha,\beta}f = \frac{\partial C}{\partial t} + \alpha(t)x + \beta(t)$$

We assume that $\lim_{x\to+\infty}\frac{\partial C}{\partial t} = 0$[2]. Then the two integration constants, α and β, are actually zero because the lower limit of the LHS as x goes to infinity is 0[3]. Thus, $\frac{1}{2}b^2 f = \frac{\partial C}{\partial t}$ is the only possible candidate. Remembering that $f = \frac{\partial^2 C}{\partial x^2}$, we get:

$$\frac{1}{2}b^2\frac{\partial^2 C}{\partial x^2} = \frac{\partial C}{\partial t} \qquad (3.1)$$

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[2] This is somewhat reasonable since $\lim_{x\to+\infty} C = 0$.

[3] Otherwise, there would be a strictly positive real γ bounding from below $b^2 f$ which is in turn lesser than $\nu^2 x^2 f$ for a non-negative ν due to the slow growth assumption, so $xf \geq \frac{\gamma}{\nu^2 x}$ which contradicts the fact that f has a finite expectation (equal to x_0).

Both derivatives are positive by arbitrage (butterfly for the convexity and conversion for the maturity). The definite candidate is then (we may impose it is positive)

$$b(x,t) = \sqrt{\frac{2\dfrac{\partial C}{\partial t}(x,t)}{\dfrac{\partial^2 C}{\partial x^2}(x,t)}} \qquad (3.2)$$

To ensure it is admissible (satisfies the slow growth condition), we impose: $\dfrac{\partial C}{\partial t} \leq x^2 \dfrac{\partial^2 C}{\partial x^2}$ for large x. This condition makes sense: diffusions cannot generate everything. To see a counterexample, let us consider a diffusion process with a binary (martingale) jump at a fixed time t^*. The Call prices it generates will increase sharply at time t^* and cannot be reproduced by a diffusion, such kinks must be ruled out.

Going back to the spot process, we indeed obtain the instantaneous volatility by

$$\sigma(S,t) = \frac{b(S,t)}{S}$$

Reminding that x actually denotes the strike, we can rewrite (3.1) as:

$$\frac{1}{2} b^2 \frac{\partial^2 C}{\partial K^2} = \frac{\partial C}{\partial T}$$

This equation has the same flavour as, but is distinct from, the classical Black-Scholes partial differential equation which involves, for a fixed option, derivatives with respect to the current time and value of the spot.

4. Discretization

It is indeed possible to compute numerically b from the relation (3.2) obtained from the continuous time and price analysis, and to discretize the associated spot process with explicit recombining binomial (Nelson & Ramaswamy [15]) or trinomial (Hull & White [9]) schemes. We prefer however to present a construction which makes use of a new technique widely used for interest rate model fitting: forward induction (Jamshidian [12] and Hull & White [10]).

It is worthwhile stressing the following point: it is actually quite easy to find a set of coefficients which correctly prices options since degrees of freedom are in superabundance compared to the constraints. The situation is analogous to the one encountered in the continuous case where various diffusions could generate the same densities. However, imposing the martingale condition (risk-neutrality) leads to uniqueness. In the discrete time setting, the martingale condition expressed at each node, gives additional constraints. This extra structure is a key point in our pricing/hedging approach but existence and uniqueness are in general not achieved by a simplistic discretization. The trinomial one nicely meets these requirements.

We build a trinomial tree with equally spaced time steps and a price step consistent with the highest volatility. Weights will be assigned to the connections, which will allow to compute the discounted probability of each path, hence to value any path-dependent option. It is actually possible to reduce the complexity of the computation in many cases.

At each discrete date, all profiles consisting of continuous piecewise linear functions with break points located at inner nodes of the tree are asked to be correctly priced by the tree. At the n^{th} step, the aforementioned space is of dimension $2n+1$, for any such profile is uniquely characterized by the value it takes on the $2n+1$ nodes. It contains the zero-coupon, the asset itself and all Calls (and Puts) whose strikes are the inner nodes. To each node we associate an *Arrow-Debreu* profile whose value is 1 on this node and 0 on the others.

A node is labelled (n,i) with n denoting the time step and i the price step. Its associated Arrow-Debreu price is noted $A(n,i)$ and the weight of the connection between nodes (n,i) and $(n+1,j)$, $j = i-1$, i, or $i+1$ is noted $w(n,i,j)$. The weights are computed through the tree in a forward fashion.

We can exploit two types of relations:

(1) Forward relations, which relate the Arrow-Debreu price of a node to the Arrow-Debreu prices of its immediate predecessors.

(2) Standard backward relations, which link the value of a contingent claim at a node to its value at the immediate successors. We apply this relation to two simple claims: a unit of the numeraire and one unit of the spot, both to be received one time step later.

The generic step of the algorithm is as follows:

Compute $w(n,i,i-1)$ from $A(n+1,i-1)$, $A(n,i)$, $A(n,i-1)$, $A(n,i-2)$, $w(n,i-1,i-1)$ and $w(n,i-2,i-1)$.

Compute $w(n,i,i)$ and $w(n,i,i+1)$ from the forward discount factors of the cash and the spot.

5. Hedging

The knowledge of the whole process allows for the pricing of path-dependent options (by Monte-Carlo methods) and American options (by Dynamic Programming). It also allows for the hedging through an equivalent spot position because the sensitivity of the options with respect to the spot can be computed: knowing the full process, it is possible to shift the initial value and to infer the process which starts from this new value and the new price it incurs. Delta hedging can then be achieved, which will be effective throughout the life of the option if the spot behaves according to the inferred process.

It probably will not, which leads us to a more sophisticated method of hedging. We can build a robust hedge which will be efficient even if the spot does not behave according to the instantaneous inferred volatilities of the diffusion process.

The idea is to associate to every contingent claim X a portfolio of European options which will be tangent to it in the sense that it will change in value identically up to the first order for changes in the volatility manifold $\sigma(K,T)_{K,T}$.

We proceed as follows:
A local move of the volatility manifold around (K_0,T_0) will lead to a new diffusion process, hence to a new value of X. We can then compute the sensitivity of X to a change of volatility $\sigma(K_0,T_0)$ and the equivalent $C(K_0,T_0)$ position. Repeating for all (K,T), we obtain a spectrum of sensitivities $Vega(K,T)_{K,T}$ and the associated (continuous) portfolio of $C(K,T)$, which can be seen as a projection of X on the $C(K,T)$. This portfolio will behave up to the first order as X, even if the market evolves transgressing the induced forward volatilities computed above.

6. Conclusion

The contribution of this paper is twofold:

On the theoretical side, it shows that under certain conditions it is possible to recover from the conditional laws a full diffusion process whose drift is imposed. It means that from option prices observed in the market we can induce a unique diffusion process.

On the practical side, it tells how to elaborate a sound pricing for path-dependent and American options. Moreover, it finely assesses the risk of such options by performing a risk analysis along both strikes and maturities. This enables rightly the full integration of these options in a book of standard European options, which is clearly a key point for many financial institutions.

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