Homework 3

Pricing and Hedging

Consider a stock S_t in the Black-Scholes model:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

starting at $S_0 = 100$. For $\sigma = 20\%$ per year, and starting with $r = 0\%$ per year.

Denote by $C_{K,T}$ to be the price of a European call with strike K that matures at time $T = 1$ year. In other words, the payoff is $(S_T - K)^+$. Start with $K = 100$.

1. Compute the price $C_{K,T}$ by a Monte-Carlo simulation, as well as its variance. Try it for various timesteps $dt = T/N$ (e.g. $N_t = 10, 100, 252, 1000$), and number of total paths $N = 10^k, k = 3, 4, 5$.

   In other words, denoting by S_T^i the value of S_T every simulated path i, compute

$$C_{K,T} \approx \frac{1}{N}\sum_{i=1}^{N}(S_T^i - K)^+, \quad v \approx \frac{1}{N-1}\sum_{i=1}^{N}\left((S_T^i - K)^+ - \frac{1}{N}\sum_{j=1}^{N}(S_T^j - K)^+\right)^2$$

2. Another way of simulating the price at time 0 is to use the Δ-hedging strategy provided by the Black-Scholes price.

   Recall this value Δ_t of the Δ-hedge function of time and the stock position, and justify the formula

$$(S_T - K)^+ - C_{K,T} = \int_0^T \Delta_t dS_t$$

   By discretizing the integral above, compute the price $C_{K,T}$ by a Monte-Carlo simulation;

$$C_{K,T} \approx \frac{1}{N}\sum_{i=1}^{N}\left[(S_T^i - K)^+ - \sum_{k=0}^{N_t-1}\Delta_{kdt}^i\,(S_{(k+1)dt}^i - S_{kdt}^i)\right]$$

   as well as the variance of this Monte-Carlo simulation.

3. Explain the improvement of variance by the second method, and compare with the analysis in the slides.

   Hint: What is theoretically the expectation of the Δ-hedging strategy $\int_0^T \Delta_t dS_t$? Is it highly correlated with $(S_T - K)^+$?