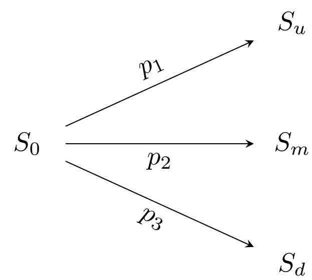
Homework 2

Exercise 1

Consider a trinomial tree, in a one period economy, that is;



For some $S_u \geq S_m \geq S_d$ and $p_i \geq 0$, $\sum_i p_i = 1$. Assume that here is an interest rate $r \geq 0$ per period, and that $S_d \leq rS_0 \leq S_u$.

(a) Does there exist a risk neutral measure? Is it unique? Exhibit one example if it exists or explain why there are no examples.

(b) Assume you have an option with payoff V_u, V_m, V_d in the corresponding situation. Is there a unique price? If so write it down. If not, what are the possible prices (give an upper and lower bound)?

(c) Assume you are in an N-nomial tree in a one period economy, and that there is N outcomes $\omega_1, ..., \omega_N$, with some probability p_i for each outcome ω_i. Let $r \geq 0$ be an interest rate which isn't chosen in a stupid way (there is no situation where it is always better to invest everything in r, or everything in S).
Consider an option V with payoff $V_1(\omega_k)$ for every outcome.

Intuition 1 for the Buyer's price:
If you are buying the option, you will of course want to do so at the lowest possible price (Buyer's price), but still at a fair price.
First, you would not like to spend more than the value of a portfolio that yields a higher payoff than the option.
So limit yourself to portfolios that 'sub-replicate' the option, that is those portfolios whose value at time 1 is almost surely less or equal than V_1. Then their value at time 0 has to be lower or equal than V_0 otherwise this would yield an arbitrage.
But not all these portfolios make sense; some of them overwhelmingly underachieve compared to the option. It is a bad idea to compare them all with V_1.
The interesting ones are the ones that are cheap, but closest to V_1 in terms of replication. So out of all cheap portfolios, they must be the most expensive ones because they yield the highest payoffs.
So the best you can do when buying the option is to be satisfied with the price of the 'most expensive cheap portfolio'.

Mathematically, the Buyer's price is defined as the value of the most expensive 'sub-replicating' portfolio, that is

$$\begin{aligned} &\max_{\Delta,c} \;\; \Delta S_0 + c \\ &s.t. \;\; \Delta S_1(\omega_k) + rc \leq V_1(\omega_k), \quad \forall k = 1,..,N \end{aligned}$$

This is a linear programming maximization problem with linear constraints. Use appropriate Lagrange multipliers λ_k to write it as a max/min unconstrained problem;

$$\max_{\Delta,c} \min_{\lambda_k \geq 0} ...$$

(fill in the dots).
Switch the max/min to min/max; the inner maximization problem should simplify and yield some constraints. Write the simplified problem as

$$\begin{aligned} &\min_{\lambda_k \geq 0} ... \\ &\text{subject to the constraints ...} \end{aligned}$$

(fill in the dots).
Set $q_k = r\lambda_k$ and rewrite the problem in terms of q to get:

$$\begin{aligned} \min_{q_k \geq 0} &\sum_k \frac{1}{r} V_1(\omega_k) q_k \\ \text{s.t} &\sum_k \frac{1}{r} S_1(\omega_k) q_k = S_0 \\ &\sum_k q_k = 1 \end{aligned}$$

Remark. Notice that the constraints on q ensure that it is defined as a risk neutral measure; if we write all the sums as $\mathbb{E}_q$, we get:

$$\begin{aligned} \min_{q \text{ probability measure}} \; &\mathbb{E}_q \left[\frac{1}{r} V_1 \right] \\ \text{s.t } &\mathbb{E}_q \left[\frac{1}{r} S_1 \right] = S_0 \end{aligned}$$

This formulation is a clearer definition of the Buyer's price: it is the cheapest price that you get out of all the prices computed by all risk neutral measures q compatible with the stock dynamics S.

(d) Do the exact same thing with the Seller's price:

$$\begin{aligned} &\min_{\Delta,c} \;\; \Delta S_0 + c \\ &s.t. \;\; \Delta S(\omega_k) + rc \geq V(\omega_k), \quad \forall k = 1,..,N \end{aligned}$$

Exercise 2

Assume that we are in the Black-Scholes setting, that is the stock price is given in the risk neutral measure by:

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t$$

with some interst rate $r > 0$ and volatility $\sigma > 0$, and $\widetilde{W}_t$ a Brownian motion in the risk measure. Assume that today's price $S_t = s > 0$.

(a) Using the Black-Scholes formula for the price of a European Call, give an analytical formula for the price of a Bull call spread which payoff is given by

$$V(S_T) = \begin{cases} B, \text{ if } S_T > B \\ \frac{B+A}{B-A} S_T - \frac{2AB}{B-A}, \text{ if } S_T \in [A, B] \\ -A, \text{ if } S_T < A \end{cases}$$

for some $0 < A < B$.

(b) Using the Black-Scholes formula for the price of a European Call, give an analytical formula for the price of a Butterfly spread which payoff is given by

$$V(S_T) = \begin{cases} 0, \text{ if } S_T < K - \delta \\ \frac{1}{\delta}(S_T - (K - \delta)), \text{ if } S_T \in [K - \delta, K) \\ -\frac{1}{\delta}(S_T - (K + \delta)), \text{ if } S_T \in [K, K + \delta] \\ 0, \text{ if } S_T > K + \delta \end{cases}$$

for some $K, \delta > 0$.

Hint: Can these payoffs be replicated with a combination of calls?

Exercise 3

In this exercise S_t represents the price of a stock. It is assumed that S_t has a continuous trajectory.
We would like to price a down-and-out call; given a maturity time T, a strike K and a barrier B, this option has a payoff $(S_T - K)_+$ only if $S_u \geq B$ for all $u \in (t, T)$. If it ever hits the barrier B before maturity, the payoff is 0.
The payoff is thus given by

$$(S_T - K)_+ \mathbb{1}_{S_u \geq B, \forall u \in (t,T)}$$

In this exercise we would like to price the above option.

Using replication arguments

(a) In this question we do **not** assume the dynamics of the stock price; in particular we do **not** know if it follows a Geometric Brownian Motion or any other type of Ito diffusion. We only know that the trajectories are continuous.
Assume that we start at $S_t = 100\$$, and that the barrier and strike are the same $B = K = 80\$$. Also assume that the interest rate $r = 0$ for simplicity.
Find a price V_t of the option by constructing a replicating portfolio. Your reasoning should only use simple no arbitrage arguments and the hedging strategy should be very simple.
Would this reasoning still work if $K \neq B$?

Let's return to the general case; K, B are general and might not be the same, the constant interest rate r is not necessarily 0, S_t starts at $x > B$ and follows a Geometric Brownian Motion

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t$$

The following questions will present the two main numerical approaches to price options.

In all the numerical experiments, we will take $t = 0$, $T = 1$ year, $r = 0.02$ (2% per year) and $\sigma = 0.15$ (15% per year), $x = 100\$$, $B = 80\$$ and $K = 110\$$.

Using Monte Carlo

(b) Use Monte-Carlo simulations to estimate the price:

$$v_t = e^{-r(T-t)} \widetilde{\mathbb{E}}[(S_T - K)_+ \mathbb{1}_{S_u \geq B, \forall u \in (t,T)} | \mathcal{F}_t]$$

Using the PDE method

The payoff of the option can equivalently be described by

$$\phi(S_\tau, \tau)$$

for

$$\tau = \min\{\inf\{u \geq t | S_u = B\}, T\}$$

and

$$\phi(y, s) = \begin{cases} (y-K)_+ \text{ if } s = T \\ 0 \text{ if } s < T \end{cases}$$

(c) Use the Feynman-Kac theorem to show that the price of the option at time t given that $S_t = x$ solves the PDE:

$$\begin{aligned} &v_t(t,x) + rxv_x(t,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) - rv(t,x) = 0, \quad \text{for } t < T \text{ and } B < x \\ &v(T,x) = (x-K)_+, \quad \text{for } B < x \\ &v(t,B) = 0, \quad \text{for } t < T \end{aligned}$$

(d) We technically need a growth condition for this PDE to make sense. We will decide that it is

$$\lim_{x \to +\infty} v(t,x) - (x - e^{-r(T-t)}K) = 0$$

Explain that growth condition.
Hint: If we start very far from the barrier, how likely are we to touch it? In this case, what would the payoff most likely be?

Numerical solution of the PDE

We will solve the PDE for $x \in (B, R)$ for some big constant $R = 300$. Let's choose $N_x = 500$ to be the number of points of x and $N_t = 252$ to be the number of time points.
Define $\Delta x = \frac{R-B}{N_x}$, $\Delta t = \frac{T-t}{N_t}$, $x_k = B + j\Delta x$ for $j = 0, .., N_x$ and $t_j = t + j\Delta t$ for $j = 0, .., N_t$.
In the following, we will refer by $u_j^k = u(t_j, x_k)$ for any function u.

(e) Explain why the final time and boundary conditions of the PDE can be numerically approximated as

$$v_{N_t}^k = (x_k - K)_+, \quad \text{for } k = 0, .., N_x \tag{1}$$

$$v_j^0 = 0, \quad \text{for } j = 0, .., N_t \tag{2}$$

$$v_j^{N_x} = R - e^{-r(T-t_j)}K, \quad \text{for } j = 0, .., N_t \tag{3}$$

(f) Use an *implicit* scheme to discretize the PDE, and write it in the form:

$$v_j^k = m_{k,k} v_{j-1}^k + m_{k,k+1} v_{j-1}^{k+1} + m_{k,k-1} v_{j-1}^{k-1}$$

(for all $k = 1, .., N_x - 1$ and $j = 1, .., N_t$), for some coefficients $m_{k,k}, m_{k,k+1}, m_{k,k-1}$ to be determined.

(g) Define the $(N_x + 1) \times 1$ vector

$$V_j = \begin{pmatrix} v_j^0 \\ \vdots \\ v_j^{N_x} \end{pmatrix}$$

and the $(N_x - 1) \times (N_x - 1)$ matrix $\tilde{M}$

$$\tilde{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & & & & \\ m_{2,1} & m_{2,2} & m_{2,3} & & & \\ & \ddots & \ddots & \ddots & & \\ & & m_{k,k-1} & m_{k,k} & m_{k,k+1} & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

So all entries of $\tilde{M}$ are 0's except $\tilde{M}_{k,k-1} = m_{k,k-1}$ for $k = 2, .., N_x - 1$,$\tilde{M}_{k,k} = m_{k,k}$ for $k = 1, .., N_x - 1$ and $\tilde{M}_{k,k+1} = m_{k,k+1}$ for for $k = 1, .., N_x - 2$.
Define the $(N_x + 1) \times (N_x + 1)$ matrix M to be

$$M = \begin{pmatrix} 1 & \cdots & & \cdots & 0 \\ m_{1,0} & & & & \\ & & \tilde{M} & & \\ & & & & m_{N_x-1,N_x} \\ 0 & \cdots & & \cdots & 1 \end{pmatrix}$$

so all entries of M are zeros except the entry $M_{2,1} = m_{1,0}$, the entry $M_{N_x,N_x+1} = m_{N_x-1,N_x}$ and all entries of M for lines 2 to N_x and columns 2 to N_x which are replaced by the entries of $\tilde{M}$.

Solve the discretized PDE by doing:

- Step 0 (Initialization): Set V_{N_t} by the final time condition (1) and set $j = N_t$
- Step 1 (Backward induction step): Set $V_{j-1} = M^{-1} V_j$
- Step 2 (Boundary values step): Set $V_{j-1}[N_x] = R - e^{-r(T-t_j)} K$ (boundary condition (3)). $V_{j-1}[0]$ should already be 0 (boundary condition (2)) but you can also set it again for security reasons if you want.
- Step 3 (Repeat): Set $j = j - 1$. If $j = 0$ stop. Else go back to Step 1.

(h) Deduce the price of the option for $x = 100$ and $t = 0$ (choose the closest grid point to 100).

Analytical solution of the PDE

(i) If we solved the PDE analytically instead of numerically, we would have obtained the formula:

$$v(t,x) = c_K(t,x) - \left(\frac{x}{B}\right)^{2\alpha} c_K\left(\frac{B^2}{x}, t\right)$$

where $c_K(t,x)$ is the Black-Scholes value of a European call of strike K and maturity T if $S_t = x$ and if the interest rate is r, and $\alpha = \frac{1}{2}(1 - \frac{2r}{\sigma^2})$. Check that this formula solves the PDE and the boundary/final time conditions.

(j) Compare the three methods (Monte Carlo, numerical solution of the PDE, analytical solution of the PDE), and explain their advantages and shortfalls.