Homework 1

Exercise 1

Let $(W_t)_{t\geq 0}$ be a standard Brownian motion, and $T > 0$.
Compute

$$\mathbb{P}(W_{T/2} < 0, W_T > 0)$$

Hint: Consider the increments $X = W_{T/2} - W_0$ *and* $Y = W_T - W_{T/2}$

Exercise 2

Let $(W_t)_{t\geq 0}$ be a standard Brownian motion, $a < b \in \mathbb{R}$ and $x \in [a, b]$. Define $X_t = x + W_t$ and

$$\tau = \inf\{t > 0 | X_t \notin (a, b)\}$$

Assume $\mathbb{E}[\tau] < +\infty$.
We would like to compute

$$p_x = \mathbb{P}(X_\tau = b)$$

which is the probability that the Brownian particle exits the interval (a, b) at b, given that it started at x.

(a) Probability method: Using Doob's optional stopping time theorem, show that

$$\mathbb{E}[X_\tau] = x$$

Deduce that

$$p_x = \frac{x - a}{b - a}$$

(b) PDE method: By noticing that

$$p_x = \mathbb{E}[\mathbb{1}_{X_\tau = b} | X_0 = x]$$

write a PDE (in this case ODE since $x \in \mathbb{R}$) satisfied by p_x, as well as the required boundary conditions. Solve the PDE and show that

$$p_x = \frac{x - a}{b - a}$$

(c) Random walk method: Let $N \in \mathbb{N}^*$ and for $k \in \mathbb{Z}$, define

$$x_k = a + k\frac{b - a}{N}$$

Consider a symmetric random walk on the x_k's, that is a discrete time stochastic process X_n^N such that:

$$\mathbb{P}(X_{n+1}^N = x_j | X_n^N = x_i) = \frac{1}{2}(\mathbb{1}_{j=i+1} + \mathbb{1}_{j=i-1})$$

i.e. this process has 1/2 chance of going left or right.
Assume that $X_0^N = x_k$ for some $k \in \{0, ..., N\}$, and define

$$\tau^N = \inf\{n \geq 0 | X_n^N \notin (a, b)\}$$

Let $p_{x_k} = \mathbb{P}(X_\tau^N = b | X_0^N = x_k)$.

Find a recursion relationship between $p_{x_k}, p_{x_{k-1}}$ and $p_{x_{k+1}}$ (as well as boundary conditions for $k = 0, N$), and solve it to show that

$$p_{x_k} = \frac{k}{N} = \frac{x_k - a}{b - a}$$

Not part of the homework: Can you solve the same question if the random walk was biased, i.e. a probability α to go right and $1-\alpha$ to go left?

Exercise 3

(a) Download last year's daily close price of Google, Amazon, Facebook, Apple and Microsoft. You can find the data for example in the Yahoo Finance website (and can choose any class of the stocks if there are multiple ones).

(b) Assume that company i has its stock price following a geometric Brownian motion

$$dX_t^i = \mu^i X_t^i dt + \sigma^i X_t^i dW_t^i$$

with $d < W^i, W^j >_t = \rho_{ij} dt$ for $i \neq j$ being the instantaneous correlation between the random drivers of each stock, $\mu^i \in \mathbb{R}$ is the drift and $\sigma^i \in \mathbb{R}_+$ is the volatility (all these constants are assumed independent of time).
Using Ito's lemma, show that

$$\begin{cases} \int_0^T d < \log X^i >_t = (\sigma^i)^2 T \\ \int_0^T d < \log X^i, \log X^j >_t = \sigma^i \sigma^j \rho_{ij} T \end{cases}$$

Thus, one way to estimate σ^i then ρ_{ij} is by doing:

$$\begin{cases} \sum_{t=1}^T \log \left(\frac{X_t^i}{X_{t-1}^i} \right)^2 \approx (\sigma^i)^2 T \\ \sum_{t=1}^T \log \left(\frac{X_t^i}{X_{t-1}^i} \right) \log \left(\frac{X_t^j}{X_{t-1}^j} \right) \approx \sigma^i \sigma^j \rho_{ij} T \end{cases}$$

Estimate both quantities, and deduce the variance of each stock $(\sigma^i)^2$, and the correlations ρ_{ij} using the formulas above.
Help: $d < X >_t$ is sometimes written in the litterature as $(dX_t)^2$, and $d < X, Y >_t$ as $dX_t dY_t$

(c) Assume $\mu^i = 0$, $\forall i$. Using a Cholesky decomposition of the matrix containing the ρ_{ij}'s, simulate a trajectory for each stock. Plot all trajectories on the same graph.