“Differential Geometry is the study of indices.” -Folklore

The general problem in geometry is to understand geometric invariants, what structure they impose on the underlying manifold and how they transform under different representations of the manifold. A natural question is also, when the geometries are in fact the same, at least locally. These problems can be considered for any structure, classic examples being Riemannian metrics, symplectic forms, almost complex structures, Kähler metrics etc. For all of these basic questions have been pretty much solved. Here I will present the basic theory for the Riemannian metrics.

1 Riemannian metric tensor

We start with a metric tensor

\[ g_{ij}dx^i dx^j. \]

Intuition being, that given a vector with \( dx^i = v^i \), this will give the length of the vector in our geometry. We require, that the metric tensor is symmetric \( g_{ij} = g_{ji} \), or we consider only the symmetrized tensor. Also we need that \( g_{ij} \) is invertible, or non-degenerate (but not necessarily positive definite) The Euclidean metric is given by \( g_{ij} = \delta_{ij} \). The first question classically is how this tensor transforms. Say \( y(x) \) is a co-ordinate transform, then by geometric considerations of a vector we get

\[ dy^i = \frac{\partial y^i}{\partial x^j} dx^j. \]

We apply the summation convention that an index appearing once up and once down is to be summed over. Applying this to the metric tensor we obtain:

\[ g_{ij}(x) = \tilde{g}_{\mu\nu} \frac{\partial y^\mu}{\partial x^i} \frac{\partial y^\nu}{\partial x^j}. \]
This is an instance of a covariant 2-tensor. In general we could say that an arbitrary agglomeration of indexed functions $b_{r_1 \ldots r_q}^{s_1 \ldots s_p}$ is a p-covariant, q-contravariant tensor if

$$b_{r_1 \ldots r_q}^{s_1 \ldots s_p} = \tilde{b}_{\tilde{r}_1 \ldots \tilde{r}_q}^{\tilde{s}_1 \ldots \tilde{s}_p} \frac{\partial y^{\tilde{s}_1}}{\partial x^{r_1}} \ldots \frac{\partial y^{\tilde{s}_p}}{\partial x^{r_q}} \frac{\partial x^{r_1}}{\partial y^{\tilde{r}_1}} \ldots \frac{\partial x^{r_q}}{\partial y^{\tilde{r}_q}}.$$

Furthermore, the metric tensor is sometimes still referred to as the first fundamental form. The second fundamental form refers to embedded submanifolds. The notation used for tensors is useful among other reasons for the following result/principle:

**Lemma 1.** Any formula involving products and sums of co-/contra-variant tensors written following the summation convention is a co-/contravariant tensor. The summation convention: Index appears only once down or up, and if both it is to be summed over. Furthermore summation is to be understood termwise, i.e. a summation is performed only for that term. Each term in the sum should follow this convention individually, and an index being summed over in one term can't be not summed over in another term. In particular if all indices are summed over, then the object is an invariant scalar.

It is a useful exercise to check, that the inverse of the matrix $g_{ij}$, which we denote $g^{ij}$ is in fact a 2-contravariant tensor. There are instances in which formulas that a priori would not be tensors, actually become tensors. One example is the exterior product of an alternating form. This is because derivatives are not in general co-variant (as seen below). On the other hand usually they can be replaced by co-variant derivatives.

Associated to any tensor there are a myriad of other objects, that will also be tensors and somehow relate to the object discussed. There are also natural operations like tensor products, and additions of tensors of the same type. The main problem for us about the metric tensor is the following.

**Problem 1:**
Given two metric tensors $g_{ij}$ and $\tilde{g}_{\mu\nu}$, when can they be transformed into each other by a change of co-ordinates. That is when is there a $y(x)$, such that

$$g_{ij}(x) = \tilde{g}_{\mu\nu} \frac{\partial y^\mu}{\partial x^i} \frac{\partial y^\nu}{\partial x^j}.$$

In particular, when is the metric euclidean. That is the particular case when $g_{ij} = \delta_{ij}$.
One of the useful expressions in Riemannian geometry is a Christoffel symbol, defined as:

\[ \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right). \]

Note, while we require this to satisfy the summation convention, it is not a co-/contravariant tensor. Below you see why. It is confusing why this should be of interest, so let us go back to Riemanns time and simply consider the problem at hand. [I am reversing the order of the lecture here.] We attempt to solve the equation

\[ g_{ij}(x) = \tilde{g}_{\mu\nu}(y(x)) \frac{\partial y^\mu}{\partial x^i} \frac{\partial y^\nu}{\partial x^j}. \]

This requires us to find functions \( y(x), p_i^\mu \) such that

\[
\begin{aligned}
  g_{ij} &= \tilde{g}_{\mu\nu} p_i^\mu p_j^\nu \\
  \partial_i y^\mu &= p_i^\mu
\end{aligned}
\]

Obviously this doesn’t help much, since the equality in the first equation is a quadratic non-linear equation. The method to proceed from there is to differentiate it:

\[
0 = \partial_k g_{ij} - \partial_i \tilde{g}_{\mu\nu} p_k^\mu p_j^\nu - \tilde{g}_{\mu\nu} \left( \partial_k p_i^\mu p_j^\nu + p_i^\mu \partial_k p_j^\nu \right).
\]

From here we aim to solve for \( \partial_k p_i^\mu \). It is a common theme in differential geometry to cyclically permute co-ordinates. So consider the term \((kij - ijk + jki)\). And you obtain:

\[
0 = \partial_k g_{ij} - \partial_i \tilde{g}_{\mu\nu} p_k^\mu p_j^\nu - \tilde{g}_{\mu\nu} \left( \partial_k p_i^\mu p_j^\nu + p_i^\mu \partial_k p_j^\nu \right).
\]

The trick now is to observe that \( g_{\mu\nu} = g_{\nu\mu} \). Thus we can switch \( \mu \rightarrow \nu \) in any term in the parentheses. Thus we can also switch \( i \rightarrow j \) and \( \mu \rightarrow \nu \) in the second and fifth terms, using the same symmetry. Thus we get

\[
0 = \partial_k g_{ij} - \partial_i \tilde{g}_{\mu\nu} p_k^\mu p_j^\nu - \tilde{g}_{\mu\nu} \left( 2 \partial_k p_i^\mu p_j^\nu \right).
\]
Multiply by $\tilde{g}^{\nu \lambda}$, divide by two, multiply by inverse of $p_j^\nu$ and use the contravariance of the inverse gives:

$$\partial_k p_i^j = \Gamma_k^i_{\lambda} p_j^\lambda - \tilde{\Gamma}^j_{\mu \nu} p_i^\mu p_j^\nu.$$ 

Thus we are lead to the equations:

$$\begin{cases} 
\partial_i y^\mu = p_i^\mu \\
\partial_k p_i^j = \Gamma_k^i_{\lambda} p_j^\lambda - \tilde{\Gamma}^j_{\mu \nu} p_i^\mu p_j^\nu
\end{cases}$$

We need some initial conditions $(y_0, p_0)$, such that $g_{ij} = \tilde{g}_{\mu \nu} p_i^\mu p_j^\nu$. A remark is in order. On the one hand we used the assumption that our map preserves the metric tensor to derive these equations. On the other hand, if a solution exists to the given equation and satisfies this initial condition, then it will preserve the metric tensor. This follows by backtracking the previous calculations to see that the derivative of the difference $g_{ij} - \tilde{g}_{\mu \nu} p_i^\mu p_j^\nu$ vanishes.

The second is just linear algebra, and $y_0$ in our case can be assumed arbitrary and fixed. Note the philosophy: Linear algebraically all symmetric two tensors are interchangable by a change of basis (assuming signature of the matrix, that is the number of negative eigenvalues and positive ones, is fixed). Now we are asking whether this can be done at every point in a smooth way. That is pointwise it is easy to change co-ordinates so that the metric tensor is whatever you want, but globally it is harder and in fact there is an obstruction.

This system is a quasi-linear partial differential equation. These systems were considered already in the 19th century and were well understood already then. In order for the system to be solvable the mixed partials must equal, by elementary calculus. This lengthy calculation leads to two equations:

$$\begin{cases} 
\Gamma_k^i_{\lambda} p_j^\lambda - \tilde{\Gamma}^j_{\mu \nu} p_i^\mu p_j^\nu = \Gamma_k^i_{\lambda} p_j^\lambda - \tilde{\Gamma}^j_{\nu \mu} p_i^\nu p_j^\mu \\
R^i_{jk} p_k^\lambda = R^i_{\mu \nu} p_j^\nu p_j^\mu
\end{cases}$$

The first is clearly a result of the definition of Christoffel symbols and the obvious symmetry in them. Here we define the Riemannian curvature tensor:

$$R^i_{jk} = \partial_i \Gamma^i_{jk} - \partial_j \Gamma^i_{ik} + \Gamma^s_{ik} \Gamma^i_{js} - \Gamma^s_{jk} \Gamma^i_{is}.$$ 

Often one considers various different ways of writing this which amount to “raising” or “lowering” an index. If $b^i_{\alpha_1 \ldots \alpha_p}$ is a tensor, we lower an index by
Similarly one raises an index. Since these arise from the same geometrical object, i.e. tensor, one denotes them by the same symbol, but just either places the corresponding indices down or up. Furthermore a useful operation is a contraction, in which one sums over a lower index and higher index. For example

\[ Ric_{jk} = R^i_{ijk}. \]

By the general principle for tensors alluded to above, all these follow the summation convention, and thus define co-variant and contravariant tensors in natural ways. This particular example is the Ricci tensor. There are many conventions for the orders of the indices in the Riemannian tensor, and thus some of them will give another sign for our objects of interest. By calculations one can prove:

\[
\begin{align*}
R_{ijkl} &= -R_{jikl} \\
R_{ijkl} + R_{sklj} + R_{sljk} = 0 \\
R_{ijkl} &= R_{klij}
\end{align*}
\]

The second of these is referred to as the first Biachi Identity. There is a second Biachi identity, concerning covariant derivatives of the Riemannian tensor, but it isn’t relevant now, and I haven’t even defined covariant derivatives yet.

Thus our equation

\[
\begin{align*}
\partial_i y^\mu &= p_i^\mu \\
\partial_k p_i^\mu &= \Gamma^\lambda_{ki} p_j^\mu - \tilde{\Gamma}^\mu_{ij} p_k^\nu
\end{align*}
\]

has a solution, if and only if the integrability conditions are satisfied. This is a way of proving in fact, that the Riemannian tensor is in fact a tensor. Namely, say we have a co-ordinate transform of the metric. Then it is a solution to the PDE given above, and furthermore it then must satisfy the integrability conditions. But that merely states that the curvature tensor is a 3-covariant, 1-contravariant tensor. In general the equation involving a four-tensor, this imposes additional constraints on the problem. That means, that in order to solve the problem, you would differentiate the equality

\[ R^l_{ijk} p^\mu_l = \tilde{R}^\lambda_{\muνκ} p^\mu_i p^\nu_j p^κ_k \]

and derive new conditions for \( \partial_k p^\mu_i \) and proceed to solve these quantities in terms of others of the same type, and continue until you would end up
in an integrable equation or alternatively notice, that there are too many constraints. Obviously, this can’t be done in general. But the case of constant curvature can be done in general, and will be discussed after the integrability, in order to enhance clarity.

2 Integrability of a system

We now consider a general problem of the form

$$\sum_i A_{ji}(x)\partial_i u(x) = f_i(u(x)) \quad j = 1 \ldots k.$$ 

This is a very general problem as stated, and there are many aspects to consider. First of all, this system of $k$ equations might be overdetermined if $k > n$ (dimension of space). On the other hand it might be underdetermined, whence the hope would be to assign some initial conditions on a non-characteristic surface. These two conditions resemble more linear algebra, than analysis and isn’t really our interest here. Thus we can restrict to a case $k = n$ and the vectors $[A_{ji}]$ should form a non-degenerate matrix.

The problem is still, that the equations are not independent, since $\partial_{ij}u = \partial_{ji}u$, at least if the solution ought to be $C^2$ (which we might as well assume, since the solutions will be constructed by ODE’s and regularity theory for those should give it). To simplify the equation, and since the matrix $A_{ij}$ is invertible, we can solve the system as, where the $f$ are some new functions:

$$\partial_i u(x) = f_i(x, u(x)) \quad j = 1 \ldots n.$$ 

Plugging the previous condition in, we obtain

$$\partial_i \partial_j u^k = \sum_{s=1}^m \partial_s f_j^k(x, u) \partial_i u^s + \partial_i f_j^k$$

$$= \sum_{s=1}^m \partial_s f_j^k(x, u) f_s^k(x, u) + \partial_i f_j^k = \sum_{s=1}^m \partial_s f_i^k(x, u) f_j^k(x, u) + \partial_j f_i^k$$

$$= \partial_j \partial_i u^k.$$ 

To simplify the notation and confuse the reader, the whether the derivatives are with respect to $x$’s or $u$’s is possible to see from the corresponding
summation. Note, that this all is vector-valued. This is the integrability condition for the equation. As discussed, if there exists a $C^2$ solution, this is necessary. A mildly surprising conclusion is that it is also sufficient. We give two sketches of arguments for this.

Consider the vector field in $\mathbb{R}^n \times \mathbb{R}^m$, where $u : \mathbb{R}^n \to \mathbb{R}^m$, and consider $A_i = (e_i, f_i(x, y))$. If there is a solution $u$, then its graph $\{(x, u(x))\}$ will be tangent to the distribution defined by $A(i)$. Thus the problem becomes to determine, when the distribution of tangent spaces defined by $A_i$ admits a foliation. This is the content of the Frobenius theorem and concerns the brackets of $A_i$, that is: $[A_i, A_j] = c_{ij}^k A_k$, i.e. the span of $A_i$ is a sub-Lie algebra. A subtlety occurs, when the distribution is not integrable in the entire space, but only on a subspace, which is the result considered above with constant sectional curvature. But this works in the same framework, since we can simply treat the problem of constructing the graph in the subspace where the vectorfields are integrable. The details are left handwavy.

If this holds, the graph can be constructed by flowing out $(x_0, u_0)$ by the vector fields $A_i$, the integrability guarantees that the curves actually trace a surface. The proof of Frobenius amounts to this. A more elementary approach was discussed in class, where instead one first constructs a tentative solution by solving $u(t,x)$, which we denote $\tilde{u}(t,x)$, for every (fixed, but varying) $0 \neq x \in \mathbb{R}^n$, by considering the ODE

$$\partial_t \tilde{u}(t,x) = \sum_{l=1}^n x^l f_i (tx, \tilde{u}(t,x)), \quad u(0,x) = u_0.$$ 

For this equation a solution exists without regard to any integrability, and I shall call it the presumptive solution. Already Euler knew that these could be solved, and presumably he would be aware of some regularity questions. Nowadays we just plainly refer to the existence and uniqueness theorems that are taught to us. The question is if this actually solves the PDE. Stability (established for example by Gronwall’s inequalities) implies:
\[ \partial_t (\partial_i \tilde{u}^k(t,x) - t f_i^k(tx, \tilde{u}(t,x))) = \partial_i \partial_t \tilde{u}^k(t,x) - f_i^k - t \left( \sum_{j=1}^{m} \partial_j f_i^k(tx, \tilde{u}) \partial_t \tilde{u}^j(t,x) \right) - t \sum_{l=1}^{n} \partial_l f_i^k(tx, \tilde{u}) x^l \]

\[ = f_i^k(tx, \tilde{u}(t,x)) - f_i^k + \sum_{l=1}^{n} x^l \left[ \sum_{j=1}^{m} \partial_j f_i^k(t,x, \tilde{u}) + t \partial_i f_i^k \right] - t \left( \sum_{j=1}^{m} \partial_j f_i^k(t,x, \tilde{u}) x^j \right) - t \sum_{l=1}^{n} \partial_l f_i^k(t,x, \tilde{u}) x^l \]

\[ = \sum_{l=1}^{n} \sum_{j=1}^{m} x^l \partial_j f_i^k(t,x, \tilde{u})(\partial_l \tilde{u}^j(t,x) - t f_i^j). \]

The last equality follows since we can interchange \( i \) and \( l \) in the terms that are substracted, by integrability. Well, an sceptical mind would say, this isn’t too good, but if you denote \( Y_i^k(t,x) \) is the term on the left, then in fact we have:

\[ \partial_t Y(t,x) = F(x, \tilde{u}) Y(t,x), \]

for some matrix whose coefficients can be read off from the equation above. On the other hand \( Y_i^k(0,x) = 0 \), since \( \tilde{u}(0,x) = u_0 \) is constant and \( t = 0 \). Thus by the existence and uniqueness theory of ODE’s the only solution to this equation is \( Y = 0 \). Concluding, we have, the equality that

\[ \partial_i \tilde{u}^k - t f_i^k = 0. \]

Now let \( u(x) = \tilde{u}(1,x) \), and the previous calculations verify, that it in fact is a solution. Thus we have proven the general theorem:

**Theorem 1.** The system

\[
\begin{cases}
\partial_i u^k(x) = f_i^k(x, u(x)) & j = 1 \ldots n, k = 1 \ldots m. \\
u(x_0) = u_0
\end{cases}
\]

admits a local \( C^2 \) solution, for \( f \in C^1 \) if and only if the integrability condition

\[ \sum_{s=1}^{m} \partial_s f_j^k(x, u) f_j^s(x, u) + \partial_j f_j^k = \sum_{s=1}^{m} \partial_s f_i^k(x, u) f_i^s(x, u) + \partial_j f_i^k \]

is satisfied for all \( i, j, k \), and for all \( x, u \) in some neighborhood of \((x_0, u_0)\).
Our interest is in the case of constant sectional curvature, in which case the integrability condition does not hold in a neighborhood of \((x_0, u_0)\). However, the proof did not require this. Going back, we only needed (setting \(x_0 = 0\)):

\[
\sum_{s=1}^{m} \partial_s f^k_j (tx, \tilde{u}(t, x)) f^s_i (tx, \tilde{u}(t, x)) + \partial_i f^k_j (tx, \tilde{u}(t, x)) = \\
\sum_{s=1}^{m} \partial_s f^k_i (tx, \tilde{u}(t, x)) f^s_j (tx, \tilde{u}(t, x)) + \partial_j f^k_i (tx, \tilde{u}(t, x)).
\]

Thus the integrability needs to be only satisfied along the presumptive solution \((tx, \tilde{u}(t, x))\). Thus we obtain a finer version of the theorem:

**Theorem 2.** The system

\[
\begin{align*}
\partial_i u^k(x) &= f^k_i (x, u(x)) \quad j = 1 \ldots n, k = 1 \ldots m, \\
u(0) &= u_0
\end{align*}
\]

admits a local \(C^2\) solution, for \(f \in C^1\) if and only if the integrability condition

\[
\sum_{s=1}^{m} \partial_s f^k_j (tx, \tilde{u}(tx, \tilde{u})) f^s_i (tx, \tilde{u}) + \partial_i f^k_j (tx, \tilde{u}) = \\
\sum_{s=1}^{m} \partial_s f^k_i (tx, \tilde{u}) f^s_j (tx, \tilde{u}) + \partial_j f^k_i (tx, \tilde{u})
\]

is satisfied for all \(i, j, k\), and for all \(x, t\) in some neighborhood of \((0, 0)\), where \(\tilde{u}(t, x)\) is given by the solution to the ODE:

\[
\partial_t \tilde{u}(t, x) = \sum_{i=1}^{n} x^i f^i (tx, \tilde{u}(t, x)), \quad u(0) = u_0.
\]

**Crucial Remark:** In our metric tensor case the crucial observation is, that along the corresponding presumptive solution \((\tilde{y}, \tilde{p})(t, x)\) in fact the equality \(g_{ij}(tx) = \tilde{g}_{\mu\nu}(\tilde{y}(t, x))\tilde{p}_i^\mu(t, x)\tilde{p}_j^\nu(t, x)\) is satisfied, by the computations in the first section.

### 3 Constant sectional curvature

Constant sectional curvature \((= K)\) corresponds to a particular form for the curvature tensor:

\[
R_{ijkl} = K(g_{i\alpha}g_{j\beta} - g_{i\beta}g_{j\alpha}).
\]
Now if $K = 0$, this is Euclidean, $K > 0$ this is Elliptic and $K < 0$ this is Hyperbolic geometry. Furthermore, returning to the question of integrability, if $g, \tilde{g}$ and $R, \tilde{R}$ satisfy the aforementioned equality. By the crucial remark at the end of the previous section

$$g_{ij}(t,x) = \tilde{g}_{\mu\nu}(\tilde{y}(t,x))\tilde{p}_i^\mu(t,x)\tilde{p}_j^\nu(t,x).$$

Applying this to the riemannian tensor, we obtain:

$$R_{ijkl}(t,x) = K(g_{il}(t,x)g_{jk}(t,x) - g_{ik}(t,x)g_{jl}(t,x)) = \cdots$$

$$= \tilde{R}(t_x)_{\mu\nu\lambda\kappa\lambda}(t,x)\tilde{p}_i^\mu(t,x)\tilde{p}_j^\nu(t,x)\tilde{p}_k^\kappa(t,x)\tilde{p}_l^\lambda(t,x).$$

Geometrically this corresponds to the Sphere, Euclidean space and the Hyperbola in the Minkowski space-time. The assumption of constant sectional curvature means, that

$$K(X,Y) = \frac{R(X,Y,Y,X)}{\langle X,X \rangle \langle Y,Y \rangle - \langle X,Y \rangle^2} = \text{constant} = K.$$ 

This object has a natural interpretation as the Gaussian curvature of the geodesic submanifold spanned by these vectors. But this would require a bunch of formulas and work to obtain. It measures in some sense the curvature in the directions specified by $X,Y$. Actually, in general, this function associated to the Riemannian tensor is called the sectional curvature, and it determines the tensor completely. This can be seen from using the symmetries described above.

Consider now any of the model spaces $S^n$, $\mathbb{R}^n$, $\mathbb{H}^n$, which are the sphere, euclidean space and hyperbolic space. Furthermore each of these groups admits a group of isometries, that preserve the metric and map any pair of $X,Y$ to any other pair $\tilde{X}, \tilde{Y}$ centered at any point, as long as the lengths and angles between them are the same. This implies by the tensorality of the Riemannian tensor (I acknowledge this language is mildly tautological) concluded above, that the sectional curvature of $(X,Y)$ is the same as for $\tilde{X}, \tilde{Y}$, that it is constant independent of this choice of vectors.

We can now compute the sectional curvatures of these manifolds. For the Euclidean space $K = 0$ is trivial. For $S^n$ projecting from the north pole down on the plane, we can obtain the metric tensor as proportional to

$$g_{ij} = \frac{\delta_{ij}}{(|x|^2 + 1)}.$$
From this computing the Christoffel symbols is easyish:

\[ \Gamma^k_{ij} = -\frac{x_i \delta_{jk} + x_j \delta_{ik} - x_k \delta_{ij}}{(|x|^2 + 1)}. \]

Furthermore \( \Gamma^k_{ij} = 0 \) at \( x = 0 \). Thus we can compute

\[ R_{1221}(0) = \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^1_{12} = \frac{2}{|x|^2 + 1} + \frac{x_1^2 + x_2^2}{(|x|^2 + 1)^2} \bigg|_{x=0} = 2. \]

So the sphere has constant positive curvature. The euclidean space clearly has zero curvature. So proceed onto the Hyperbolic plane and get similarly (Poincare Disk model):

\[ g_{ij} = \frac{\delta_{ij}}{1 - |x|^2}. \]

Thus the Christoffel symbols are again

\[ \Gamma^k_{ij} = \frac{x_i \delta_{jk} + x_j \delta_{ik} - x_k \delta_{ij}}{(|x|^2 + 1)}. \]

Which is just the negative of the above, so we get \( R_{1221}(0) = -2 \), finishing the calculations.

Thus any constant curvature Riemannian manifold is locally isometric to one of these (with a given radius). This naturally imposes strong topological constraints in the manifold.