PDE Spring 2016

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Review for final #2

Now we focus on second-order elliptic and parabolic PDEs.

$u_t = k\nabla^2 u + f$, $k > 0$ — heat equation, parabolic

or

$\nabla^2 u = f$ — Poisson

$\nabla^2 u = 0$ — Laplace

The easiest method of solution depends on the boundary conditions

1. Infinite domain (heat or wave equations), use Green's functions for parabolic
2. Finite domains — separation of variables

Diffusion equation on the whole real line or plane

$$\begin{cases} u_t = k u_{xx} & x \in \mathbb{R} \\ u(x,0) = \varphi(x) & \text{IC} \end{cases}$$

(no BCs!)

Solution obtained using Green's functions

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\, \varphi(y)\, dy$$

$$G(x,t) = \frac{1}{\sqrt{4\pi k t}}\, e^{-x^2/(4kt)}$$

Same works for wave equation also with a different G but because of the simple form of G it becomes identical to d'Alambert's formula.

Import: Easy to generalize to 2D/3D ③

In 2D:

$$G(x,y,t) = G(x,t)\, G(y,t)$$

(recall midterm!)

$$= \frac{1}{4\pi k t} e^{-(x^2+y^2)/(4kt)}$$

$$= \frac{1}{4\pi k t} e^{-r^2/(4kt)}$$

Formula is the same but now integral is over the whole plane:

$$u(x,y,t) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} G(x-x', y-y', t)\, \varphi(x',y')\, dx'\, dy'$$

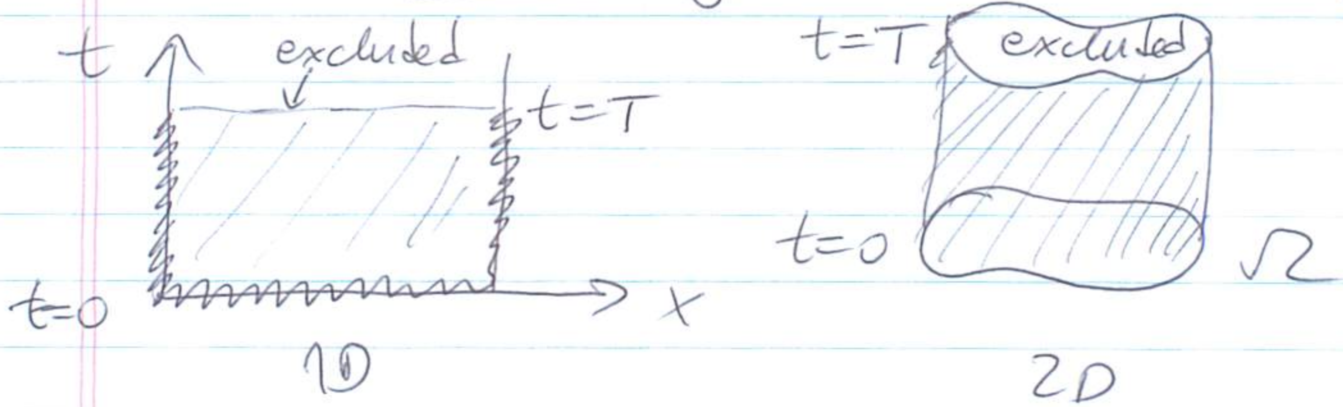
Obvious generalization to 3D/nD

Green's function difficult to figure out in general

Maximum principle

For both the heat equation and the Laplace equation, an extremum principle applies.

For heat, the extremum is achieved on one of the boundaries of the space-time domain excluding $t = T$ boundary



For Laplace, the extremum is achieved on the boundary of the domain $\partial\Omega$

Bounded domains

General problem:

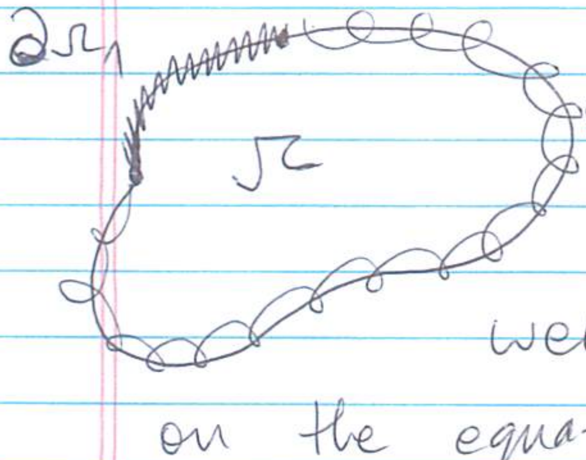
$$\begin{cases} u_t = \mathcal{L} u + f(\vec{x}, t) \quad \text{for } \vec{x} \in \Omega \\ u(\vec{x}, t=0) = u_0(\vec{x}) \quad \text{IC} \\ u(\partial\Omega_1) = \varphi(\vec{x} \in \partial\Omega_1, t) \\ \frac{\partial u}{\partial \vec{n}} \cdot \vec{n}(\partial\Omega_2) = \psi(\vec{x} \in \partial\Omega_2, t) \end{cases}$$

(none for Laplace/Poisson)

Dirichlet

Neumann

$$\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$$



Note

In order for the problem to be well-posed depending on the equation and BCs, f and g may need to satisfy additional cond.

Use superposition to split into subproblems that may be easier to solve (6)

(P1) Steady-state problem

$$\begin{cases} \mathcal{L} v = 0 \\ + \text{ BCs for } u_1 \text{ are the } \underline{\text{same}} \end{cases}$$

This handles the inhomogeneous BCs for us (so skip if homogeneous)

If φ and ψ do not depend on t, then

$v \equiv v(\vec{x})$ only depends on $\vec{x}$

In general

$v \equiv v(\vec{x}\,; t) \equiv v(\vec{x}, t)$

depends on t, but t is only a parameter for the solution since $\mathcal{L}$ only involves spatial derivatives

<u>Note</u>: In some problems, it may be simple to solve

$$\mathcal{L}\vartheta = -f \quad \text{directly}$$

If so, this will speed up the process (recall problem in homework $u'' = Q$).

P2

$$\begin{cases} w_t = \mathcal{L}w + f - \vartheta_t \\ \left.\begin{aligned} & w(\partial\Omega_1) = 0 \\ & \frac{\partial w}{\partial \vec{n}} \cdot \vec{n}(\partial\Omega_2) = 0 \end{aligned}\right\} \ \underline{\text{homogeneous}} \ \text{BCs} \\ w(\vec{x}, t=0) = u_0(\vec{x}) - \vartheta(\vec{x}, 0) \end{cases}$$

This problem we solve via the method of <u>separation of variables</u>, which here becomes the method of <u>orthogonal series</u>

$$\boxed{u = \vartheta + w} \quad \text{Superposition}$$

Note: If the original problem was not a heat equation but rather Poisson, the process would be the same

$$\begin{cases} \mathcal{L}u = f \\ + \text{BCs} \end{cases}$$

P1: $\begin{cases} \mathcal{L}v = 0 \\ + \text{BCs} \end{cases}$ P2: $\begin{cases} \mathcal{L}w = f \\ + \underline{\text{homogeneous}} \\ \text{BCs} \end{cases}$

and again $u = v + w$

For final, do not try to memorize this. I will require you to show that indeed you have used the superposition correctly, i.e., to prove that $u = v + w$ solves the original BVP

Key idea: Handle inhomogeneous BCs first, then solve the rest with homogeneous BCs

Separation of variables

Consider first solving the Laplace equation with inhomogeneous BCs

$$\begin{cases} \mathcal{L} u = 0 \\ u(\partial\Omega_1) = \varphi \\ \partial_n u(\partial\Omega_2) = \psi \end{cases}$$

If 1D it is easy to solve (ODE!)

This is solveable on a rectangular domain in 2D/3D using separation of variables

Assume separable solution

$$u = X(x)\, Y(y)$$

and plug into equation, then separate x and y pieces to get:

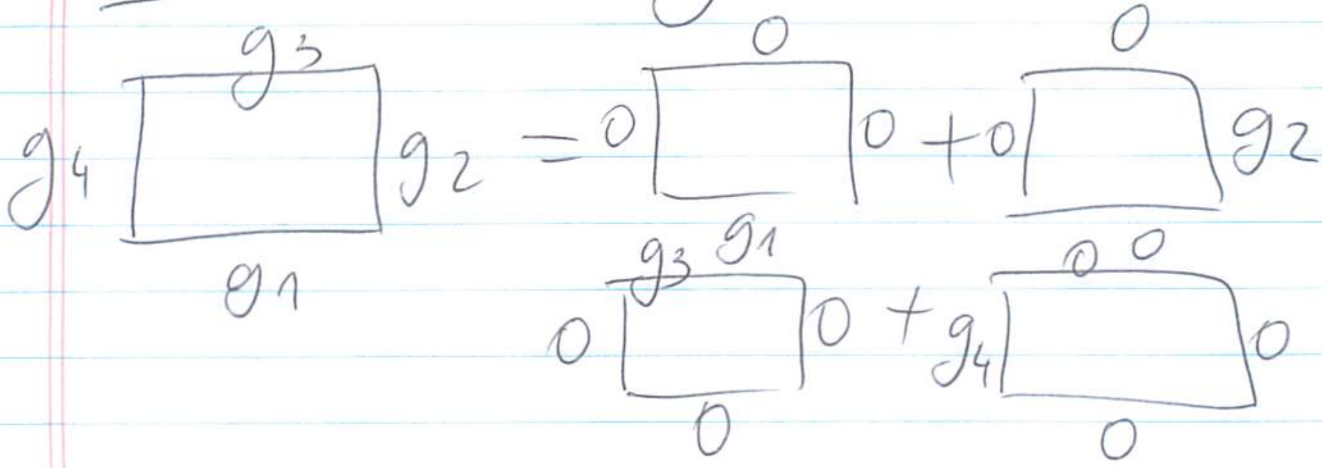
$$F(x) = G(y) = \text{const} = \lambda$$

and then solve ODEs

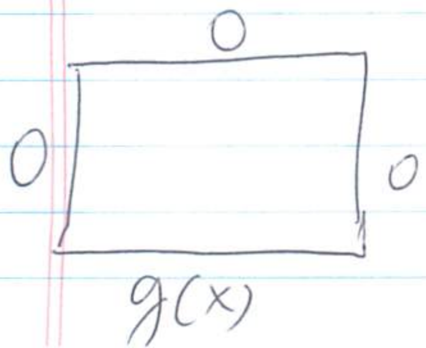
$$\begin{cases} F(x) = \lambda \\ G(y) = \lambda \end{cases}$$

One by one, starting from the simplest one. To obtain possible values of λ, use BCs as follows.

First, split problem so that inhomogeneous BC is on only <u>one boundary</u>

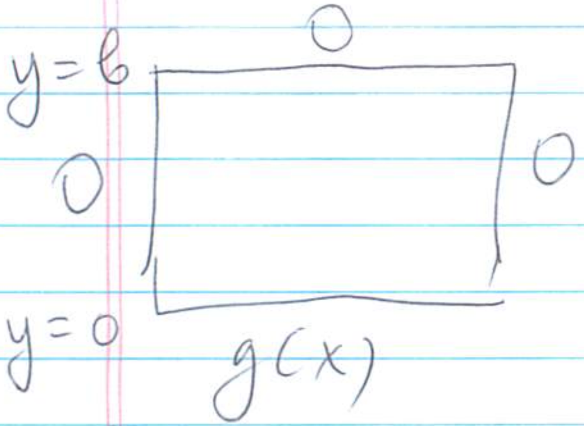


First, solve ode along homogeneous (or periodic) direction, e.g., for



solve $F(x) = \lambda$ with homogeneous BCs first this will determine λ (along with its sign)

then, solve along the other direction, putting homogeneous BC along homogeneous boundaries, and $\underline{1}$ (unity) along others



Solve

$$G(y) = \lambda$$

$$Y(0) = \underline{1}, \quad Y(b) = 0$$

Once you get $\underline{X}(x)$ and $Y(y)$, write solution as a superposition of separable solutions

$$u = \sum_n a_n X_n(x) \, Y_n(y)$$

and plug into BC, e.g.,

$$u(x, y=0) = g(x)$$

to get

$$\boxed{\sum a_n X_n(x) = g(x)}$$

which we need to solve for the coefficients a_n.

This is quite generally an orthogonal series, e.g., a Fourier series, since $X_n(x)$ were solutions of an equation of the form

$$\boxed{\mathcal{L} u = -\lambda u} + \text{homog. BCs}$$

where $\mathcal{L}$ was a self-adjoint operator. This implies that Eigenfunctions are orthogonal and eigenvalues are real

(L_2) Dot product (inner product):

$$(f, g) = \int_a^b f(x)\,\overline{g}(x)\,dx$$

or more generally

$$(f,g) = \int_{\Omega} f(\vec{x})\,\bar{g}(\vec{x})\,d\vec{x}$$ (13)

Ω ← over domain

If it is true that

$$(\bar{X}_n, \bar{X}_m) = 0 \text{ if } n \neq m$$

then the X_n's form an orthonormal basis and

$$\sum a_n X_n(x)$$

is an orthogonal series.

How to solve

$$\sum a_n X_n(x) = g(x)$$ | Multiply by $X_m(x)$

$$\left(X_m, \sum_n a_n X_n\right) = (X_m, g) =$$

$$\sum_n a_n (X_m, X_n) = a_m (X_m, X_m)$$

$\Rightarrow$

$$\boxed{a_m = \frac{(X_m, g)}{(X_m, X_m)}}$$ ✓

Examples

Fourier sine series

$$X'' = -\lambda X, \quad X(0) = X(a) = 0$$

Fourier cosine series

$$X'' = -\lambda X, \quad X'(0) = X'(a) = 0$$

Mixed series

$$X'' = -\lambda X, \quad X(0) = 0, \; X'(a) = 0$$

Periodic Boundaries

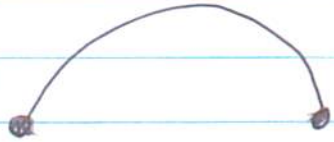
(not really boundaries – domain is topologically a circle or a torus in 2D)

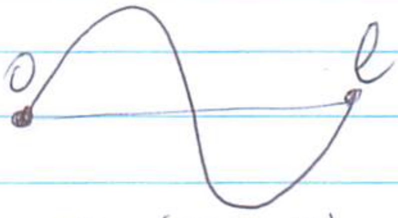
$$X'' = -\lambda X, \quad X(-\ell) = X(\ell)$$
$$X'(-\ell) = X'(\ell)$$

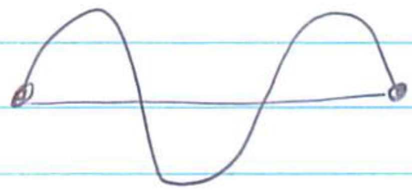
How to think about this geometrically?

We know solutions are sine and cosine

Dirichlet BCs



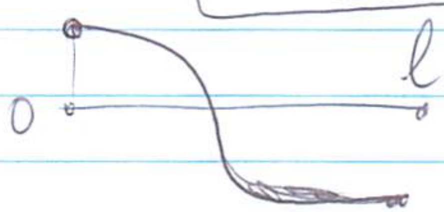


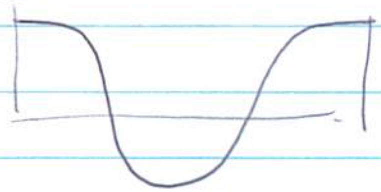


$\sin\left(\frac{\pi x}{l}\right)$ $\sin\left(\frac{2\pi x}{l}\right)$ $\sin\left(\frac{3\pi x}{l}\right)$

$\boxed{\sin\left(\frac{n\pi x}{l}\right)}$ $n = 1, 2, \ldots$

Neumann BCs



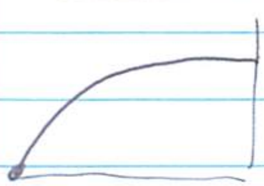


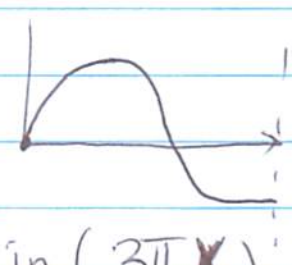
constant $= 1$ $\cos\left(\pi \frac{x}{l}\right)$ $\cos\left(\frac{2\pi x}{l}\right)$

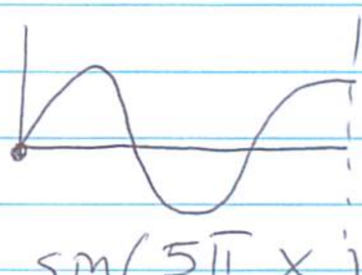
$\Rightarrow \boxed{\cos\left(\frac{n\pi x}{l}\right), \; n = 1, 2, \ldots}$

Mixed BCs

$X(0) = 0, \quad X'(l) = 0$



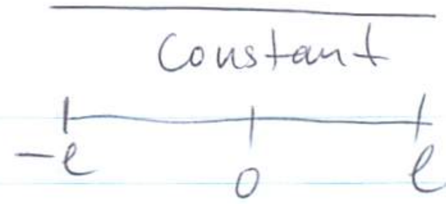


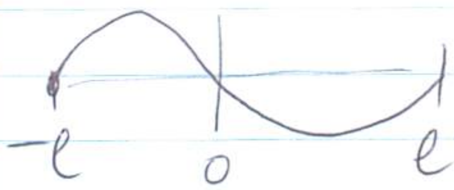


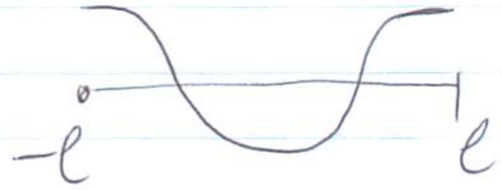
$\sin\left(\frac{\pi}{2}\frac{x}{l}\right)$ $\sin\left(\frac{3\pi}{2}\frac{\cancel{x}}{l}\right)$ $\sin\left(\frac{5\pi}{2}\frac{x}{l}\right)$

$\boxed{\sin\left((2n-1)\frac{\pi x}{2l}\right), \; n = 1, 2, \ldots}$

Periodic BCs







$\sin\left(\frac{n\pi x}{\ell}\right)$ $\qquad$ $\cos\left(\frac{n\pi x}{\ell}\right)$

Any sine or cosine that has periodicity of 2ℓ works.

Best basis for calculations is complex one:

$$\boxed{\exp\left(i\frac{n\pi x}{\ell}\right), \quad n = -\infty, \dots, 0, \dots +\infty}$$

In all cases the functions are orthogonal, so, for example

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell} = \varphi(x)$$

$$\Rightarrow \quad c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} \varphi(x) e^{-in\pi x/\ell} dx$$

since $\displaystyle\int_{-\ell}^{\ell} e^{-in\pi x/\ell} e^{in\pi x/\ell} dx = 2\ell$!

The most general case of a 1D eigenvalue problem is the Sturm-Liouville:

$$\lambda u = \mathcal{L} u = -\left(p(x)\, u'(x) \right)' + q(x)\, u$$

$p(x) > 0$ and differentiable

$q(x) > 0$ and continuous

Eigenfunctions are orthogonal

Eigenvalues are positive

$0 < \lambda_1 < \lambda_2 \ldots$

This means we can expand functions (e.g., unknown solution) as a series of such orthogonal functions.

<u>Practice</u>:
$$\begin{cases} u'' \quad u = -\lambda u \\ u(0) = u(1) = 0 \end{cases}$$

Basic idea in <u>all</u> cases:

$\left\{\right.$ Expand solution in series of eigenfunctions.

Find eigenfunctions in 2D by using separation of variables

$$u = X(x)\,Y(y)$$

to convert to two one-dimensional eigenvalue problems, which are easy to solve as ODEs.

<u>Example</u>

Heat equation with source and initial condition

$$\begin{cases} u_t = \mathcal{L}u + f \\ u(\partial\Omega) = 0 \\ u(t=0) = u_0 \end{cases}$$

$$u(\vec{x}, t) = \sum_n a_n(t)\, u_n(\vec{x})$$

If Poisson no time dependence

$$f(\vec{x},t) = \sum f_n(t)\, u_n(\vec{x}) \qquad 19$$

$$\Rightarrow \quad f_n(t) = \frac{\int_\Omega f(\vec{x},t)\, u_n(\vec{x})\, d\vec{x}}{\int_\Omega |u_n(\vec{x})|^2\, d\vec{x}}$$

Since u_n are orthogonal eigenfunctions

$$\boxed{\mathcal{L} u_n = \lambda_n u_n}$$

Plug back into PDE

$$u_t = \sum a_n' u_n$$

$$\mathcal{L} u = \sum a_n (\mathcal{L} u_n) = \sum a_n \lambda_n u_n$$

$$\Rightarrow \sum a_n' u_n = \sum a_n \lambda_n u_n + \sum f_n u_n$$

$\Rightarrow$ (since u_n linearly independent)

$$\boxed{a_n' = \lambda_n a_n + f_n(t)}$$ ODE to solve

If Poisson $\quad a_n = -f_n/\lambda_n$

This ODE can be solved using Duhamel's principle

First solve $a_n' = \lambda_n a_n \Rightarrow$

$$a_n(t) = a_n(0)\, e^{\lambda_n t}$$

and thus

$$a_n(t) = a_n(0)\, e^{\lambda_n t} + \int_0^t f_n(\tau)\, e^{\lambda_n (t-\tau)}\, d\tau$$

How to determine $a_n(0)$?

Go back to initial conditions

$$u_n(t=0) = \sum a_n(0)\, u_n = u_0$$

$$\Rightarrow \quad a_n(0) = \frac{\int_\Omega u_0\, u_n\, d\vec{x}}{\int_\Omega (u_n)^2\, d\vec{x}}$$

Same ideas just recycled over and over again!