Outline of PDE Recitations

Jiajun Tong

April 25, 2016

1 The Recitation on February 5, 2016

Key points: checking a function is a solution, dispersion relations, solving for a general solution, and others.

1. Problem 1.7 in EPDE on Page 9.
2. Problem 1.6 in EPDE on Page 9.
3. Problem 12 (a)(c)(d) in §1.1 of APDE on Page 11 (dispersion relations).
4. Dispersion relation of the damped wave equation $u_{tt} + u_t = u_{xx}$.
5. Starting from Cauchy-Riemann equation, we showed that real and imaginary parts of complex analytic functions are harmonic functions. Then comes an example $u(x, y) = \mathrm{Re}(z^3)$, the monkey saddle.
6. An example of ill-posed problem — Robin boundary condition with wrong sign.
7. A general solution of $u_{xt} + 3u_x = 1$.

2 The Recitation on February 12, 2016

Key points: first order PDE, and methods of characteristics.

1. Solution of (1.5) in §1.2 of APDE on Page 16.
2. Example 1.9 in §1.2 of APDE on Page 17.
3. Example 1.11 in §1.2 of APDE on Page 18.
4. Example 4.2 in §4.1 of EPDE on Page 39.
5. Problem 12 in §1.2 of APDE on Page 27.

3 The Recitation on February 19, 2016

Key points: revisit of what was discussed in the last recitation (first order PDE and methods of characteristics), classification of second order PDE, and solving a second order hyperbolic equation with generic coefficients.

1. Example 1.9 in §1.2 of APDE on Page 17.
2. Example 1.11 in §1.2 of APDE on Page 18.
3. Example 4.2 in §4.1 of EPDE on Page 39.
4. Problem 4.9 in EPDE on Page 54.

4 The Recitation on February 26, 2016

Key points: more about methods of characteristics, and solution of heat/diffusion equation.

1. Problem 13 in §1.2 of APDE on Page 27 ($f(x,t)$ instead of $f(x)$ is put on RHS).
2. Characteristics of the inviscid Burgers' equation $u_t + uu_x = 0$.
3. Two ways to get solution of heat equation with initial data $u(x, t = 0) = \frac{1}{\sqrt{\pi}} \exp(-(x-2)^2)$ — brute-force computation and space-time shift.

5 The Recitation on March 4, 2016

Key points: a boundary value problem of wave equation, Duhamel's formula, the δ-function.

1. Problem 4.7 in EPDE on Page 54 (also in HW 4).
2. Problem 4 in §2.5 of APDE on Page 106, and a modification of that: $u_t + 2tu_x = x\mathrm{e}^{-t}$. Note that the original problem is translation invariant in t, but the modified problem is not. As a result, when applying the Duhamel's formula, the forward map should be treated carefully in the latter case. The problem can also be solved by integration along the characteristics.
3. Some remarks on the δ-function:
   (a) It is best to understand the δ-function as a linear functional. We always define a functional by characterizing its effect on good test functions (to be more precise, C_0^∞-functions). For example, any function f in the classic sense can be realized as a functional in a natural way, i.e.,
   $$\mathcal{L}_f(g) \triangleq \int_{\mathbb{R}} f(x)g(x)\,dx, \quad \forall\, g \in C_0^\infty(\mathbb{R}).$$
   The defintion of the δ-function is given by
   $$\delta[g] = \mathcal{L}_\delta(g) \triangleq g(0), \quad \forall\, g \in C_0^\infty(\mathbb{R}).$$
   (b) Formal integral representations like $g(0) = \int_{\mathbb{R}} \delta(x)g(x)\,dx$ are helpful in understanding why the definitions concerning generalized functions are natural (e.g. recall how we define δ'), but they are just formal.
   (c) Convergence of a sequence of functions $\{f_n\}$ to the δ-function is defined as follows: we say $f_n \to \delta$ (in the sense of distribution) as $n \to +\infty$ if
   $$\lim_{n\to+\infty} \int_{\mathbb{R}} f_n(x)g(x)\,dx = g(0), \quad \forall\, g \in C_0^\infty(\mathbb{R}),$$
   or equivalently, if
   $$\lim_{n\to+\infty} \mathcal{L}_{f_n}(g) = \mathcal{L}_\delta(g), \quad \forall\, g \in C_0^\infty(\mathbb{R}).$$
4. Exercise 6.1.9 (a)(b) in P. Olver's book *Introduction to Partial Differential Equations* on Page 227.

6 The Recitation on March 11, 2016

Key points: solution of the midterm (find the solution in another PDF).

7 The Recitation on March 18, 2016

Spring break.

8 The Recitation on March 25, 2016

Key points: separation of variables.

1. 1D heat equation with (homogeneous) Neumann boundary condition.

$$\begin{aligned} &u_t = ku_{xx}, \quad x \in (0,l),\ t > 0, \\ &u(x,0) = g(x), \\ &u_x(0,t) = u_x(l,t) = 0. \end{aligned}$$

2. 1D wave equation with (homogeneous) Dirichlet boundary condition.

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \quad x \in (0,l),\ t > 0, \\ &u(x,0) = f(x), \quad u_t(x,0) = g(x), \\ &u(0,t) = u(l,t) = 0. \end{aligned}$$

3. Boundary value problem with periodic boundary condition. Find out $\lambda \in \mathbb{C}$ such that the following equation admits a non-trivial solution (and find the solution out)

$$\begin{aligned} &X''(x) = -\lambda X(x), \quad x \in (0,l), \\ &X(0) = X(l), \quad X'(0) = X'(l). \end{aligned}$$

9 The Recitation on April 1, 2016

Key points: convergence in (L^2-)norm, separation of variables for problems in spherical domains.

1. Convergence in norm (especially in L^2-norm), convergence of Fourier series in L^2-norm, Parseval's identity, and completeness of an orthogonal set (notion of orthonormal basis). It is highly suggested to go through again the last 3 pages of Prof. Donev's lecture notes Lecture 15.

2. Laplace equation in 2D unit disk with Dirichlet boundary condition.

$$\begin{aligned} &\Delta u = 0, \quad x \in B_1 \subset \mathbb{R}^2, \\ &u|_{\partial B_1}(x) = g(x). \end{aligned}$$

   As a remark, for Laplace equation in 3D, one can still do separation of variable in spherical coordinates using r, θ and ϕ, although the problem becomes more complicated. Interested students could read the section for *Laplace's spherical harmonics* on the Wikipedia page `en.wikipedia.org/wiki/Spherical_harmonics#Laplace.27s_spherical_harmonics` for a quick reference. A type of special functions called *associated Legendre polynomials* will arise in equation for θ, the zenith coordinate. See e.g. `en.wikipedia.org/wiki/Associated_Legendre_polynomials`.

3. §4.5 of APDE on Page 198. Radially symmetric solution for heat equation in 3D unit ball with homogeneous Dirichlet boundary condition.

$$\begin{aligned} &u_t = k\Delta u, \quad x \in B_1 \subset \mathbb{R}^3,\ t > 0, \\ &u|_{\partial B_1}(x,t) = 0, \\ &u(x,0) = f(x), \end{aligned}$$

   where $f(x)$ is radially symmetric.

   As a remark, for the same equation in 2D, the solutions for the r-equation are another type of special functions called *Bessel functions*. Interested students could read §4.6 of APDE.

10 The Recitation on April 8, 2016

Key points: separation of variables for problems in spherical domains, Sturm-Liouville problem.

1. Revisit of §4.5 of APDE on Page 198 — cooling of a ball. Radially symmetric solution for heat equation in 3D unit ball with homogeneous Dirichlet boundary condition.

$$
\begin{aligned}
&u_t = k\Delta u, \quad x \in B_1 \subset \mathbb{R}^3,\ t > 0,\\
&u|_{\partial B_1}(x,t) = 0,\\
&u(x,0) = f(x),
\end{aligned}
$$

   where $f(x)$ is radially symmetric.

2. Problem 5.4(a) in EPDE on Page 79 (write equations into Sturm-Liouville problem). As a remark, Sturm-Liouville problem is *not* a form that is helpful in solving the boundary value problem that arises in separation of variables — solving such type of equations usually involves some other tricks; and in many cases, the solutions can not be explicitly represented by simple functions. However, SLP admits a good theoretical framework about its eigenvalue problems (e.g. orthogonality and completeness of its eigenfunctions under some assumptions), which is crucial in doing separation of variables.

3. Example 1 in §6.4 on Page 172 in *Partial Differential Equations: An Introduction, John Wiley & Sons, any edition, ISBN-13: 978-0470054567* by Walter Strauss. Laplace equation in a 2D wedge (a sector with radius a and angle β).

$$
\begin{aligned}
&\Delta u(r,\theta) = 0, \quad r \in [0,a],\ \theta \in [0,\beta],\\
&u(r,0) = u(r,\beta) = 0,\\
&\frac{\partial u}{\partial r}(a,\theta) = h(\theta).
\end{aligned}
$$

   Note that although u is written in terms of r and θ, the Laplace operator is still defined to be $\partial_x^2 + \partial_y^2$.

11 The Recitation on April 15, 2016

Key points: use symmetry to construct solutions, wave equation on $[0,1]$ with inhomogeneous boundary data and source term.

1. Consider the heat equation on $\mathbb{R}$

$$
\partial_t v(x,t) = k\partial_{xx} v(x,t), \quad v(x,0) = \varphi(x), \quad x \in \mathbb{R},\ t > 0.
$$

   We showed that if $\varphi(x)$ is odd, $v(-x,t) = -v(x,t)$ and hence $v(0,t) = 0$; if $\varphi(x)$ is even, $v(-x,t) = v(x,t)$ and hence $v_x(0,t) = 0$. Then we use this to construct solution to the heat equation on semi-axis with homogeneous Dirichlet and Neumann boundary condition, i.e.

$$
\begin{aligned}
&\partial_t u(x,t) = k\partial_{xx} u(x,t), \quad x \in \mathbb{R}_+,\ t > 0,\\
&u(x,0) = \varphi(x), \quad u(0,t) = 0,
\end{aligned}
$$

   and

$$
\begin{aligned}
&\partial_t u(x,t) = k\partial_{xx} u(x,t), \quad x \in \mathbb{R}_+,\ t > 0,\\
&u(x,0) = \varphi(x), \quad u_x(0,t) = 0.
\end{aligned}
$$

2. Problem 9 in §4.7 of APDE on Page 215 (wave equation on $[0,1]$ with inhomogeneous boundary data and source term).

12 The Recitation on April 22, 2016

Key points: review class 1. First order transport equation on $\mathbb{R}$ (method of characteristics), heat equation on $\mathbb{R}$ (fundamental solution), wave equation on $\mathbb{R}$ (general solution, d'Alembert's formula).

1. It is adapted from problem 8 in §1.2 of APDE on Page 26. Solve

$$u_t + xu_x - 3u = t,$$
$$u(x,0) = x^2.$$

   The solution is $u(x,t) = x^2e^t + \frac{1}{9}(e^{3t} - 1) - \frac{t}{3}$.

2. Problem 4 on Page 41 in the reference book by W. Strauss. Suppose $u(x,t)$ satisfies $u_{tt} = u_{xx}$. Show that

$$u(x+h, t+k) + u(x-h, t-k) = u(x+k, t+h) + u(x-k, t-h).$$

   Hint: use the general solution $u(x,t) = F(x+t) + G(x-t)$.

3. Advection-diffusion equation. See APDE §1.4, Page 40 - Page 43. Solve

$$u_t + cu_x = ku_{xx},$$
$$u(x,0) = \varphi(x).$$

   Hint: show that $v(x,t) \triangleq u(x+ct, t)$ satisfies heat equation.