Problem 2 in the Recitation on April 15, 2016

Jiajun Tong

April 15, 2016

Problem 1 (Problem 9 in §4.7 on page 215 of APDE, with $q = 0$)**.** Solve

$$\begin{aligned} &u_{tt} = u_{xx}, \quad x \in (0,1),\, t > 0, \\ &u(0,t) = 0, \quad u(1,t) = \sin t, \quad t > 0, \\ &u(x,0) = x(1-x), \quad u_t(x,0) = 0, \quad x \in (0,1). \end{aligned} \tag{1}$$

Solution. The basic idea is to first apply a similar strategy as in the separation of variables to find out eigenfunctions in space. The eigenfunctions automatically form a basis (for $L^2([0,1])$) and thus every (L^2-)function on $[0,1]$ can be written as a series in terms of these eigenfunctions. Then it suffices to find out and solve the ODE's satisfied by the coefficients.

Before we solve for the eigenfunctions, however, we need to deal with the inhomogeneous boundary condition, since the separation of variables can only be applied to equations with homogeneous boundary conditions. (This argument may remind you of the Laplace equation in a rectangle discussed in Lecture 17, `http://cims.nyu.edu/~donev/Teaching/PDE-Spring2016/Lecture_17.pdf`, where the boundary condition on $Y(y)$ at $y = 1$ is not homogeneous. However in that case, it is only for the X-equation that we solve an eigenvalue problem; and the X-equation does enjoy homogeneous boundary conditions. It is essentially the same here — we need homogeneous boundary condition to solve for eigenvalue problems in space.)

Step 1 (**Eliminate inhomogeneous boundary condition**). Let $v(x,t)$ satisfies the following equation

$$\begin{aligned} &v_{xx}(x,t) = 0, \quad x \in (0,1),\, t > 0, \\ &v(0,t) = 0, \quad v(1,t) = \sin t, \quad t > 0. \end{aligned} \tag{2}$$

In fact, it is easy to find $v(x,t) = x \sin t$. Define $w = u - v$. Then w satisfies the following equation

$$(\partial_{tt} - \partial_{xx})w = (\partial_{tt} - \partial_{xx})u - (\partial_{tt} - \partial_{xx})v = 0 - \partial_{tt}v(x,t) = x \sin t, \tag{3}$$

together with initial and boundary conditions as follows

$$w(0,t) = u(0,t) - v(0,t) = 0, \quad w(1,t) = u(1,t) - v(1,t) = 0, \tag{4}$$

$$w(x,0) = u(x,0) - v(x,0) = x(1-x), \quad w_t(x,0) = u_t(x,0) - v_t(x,0) = -x. \tag{5}$$

(4) gives the reason why we took such boundary condition in (2) for v, while what equation v satisfies in $[0,1]$ is not really important.

Step 2 (**Find out eigenfunctions and eigenvalues in space** — similar to separation of variables). Now we obtain that w satisfies a wave equation with homogeneous boundary condition and also with a source term. What comes next is to apply a separation-of-variable-type approach to find out eigenfunctions *in space*, so that we can write everything (source term and initial data) as series of these eigenfunctions (as they form a basis).

We study the following eigenvalue problem for the differential operator in space

$$X''(x) = -\lambda X(x), \quad X(0) = X(1) = 0. \tag{6}$$

You can also start from assuming an ansatz $w_*(x,t) = X(x)T(t)$ for $w_*(x,t)$ satisfying a homogeneous wave equation without source term, $\partial_{tt} w_* = \partial_{xx} w_*$, and follow the steps in the separation of variables. It turns out there is no need to take care of the T-equation; so we will just focus on the eigenvalue problem (6) in space. The solution to (6) is given by

$$X_k(x) = \sin(k\pi x), \quad \lambda_k = k^2\pi^2, \quad k \in \mathbb{Z}_+. \tag{7}$$

They form a basis for (L^2-)function space on $[0,1]$ due to theories on Sturm-Liouville problem. Hence we can write the solution, the source terms and initial datum to be series of X_k's.

Step 3 (**Expand everything as series of eigenfunctions**). Now we assume the solution to (3) is in the form of

$$w(x,t) = \sum_{k=1}^{\infty} T_k(t) X_k(x). \tag{8}$$

Also

$$x = \sum_{k=1}^{\infty} a_k X_k(x), \quad a_k = 2\int_0^1 x \cdot \sin(k\pi x)\, dx = \frac{(-1)^{k+1}\cdot 2}{k\pi}, \tag{9}$$

$$x(1-x) = \sum_{k=1}^{\infty} b_k X_k(x), \quad b_k = 2\int_0^1 x(1-x) \cdot \sin(k\pi x)\, dx = \frac{4(1-(-1)^k)}{(k\pi)^3}. \tag{10}$$

It is a good exercise to calculate a_k's and b_k's by yourselves. Note that under this expansion, the boundary conditions (4) are naturally satisfied. By (3) and (5), we know that

$$(\partial_{tt} - \partial_{xx}) w(x,t) = \sum_{k=1}^{\infty} \left[T_k''(t) + \lambda_k T_k(t)\right] X_k(x) = \sum_{k=1}^{\infty} a_k \sin t \cdot X_k(x), \tag{11}$$

$$w(x,0) = \sum_{k=1}^{\infty} T_k(0) X_k(x) = \sum_{k=1}^{\infty} b_k X_k(x), \tag{12}$$

$$w_t(x,0) = \sum_{k=1}^{\infty} T_k'(0) X_k(x) = \sum_{k=1}^{\infty} -a_k X_k(x). \tag{13}$$

Since $X_k(x)$'s form a basis, we obtain second-order ODE's for $T_k(t)$, $\forall\, k \in \mathbb{Z}_+$,

$$T_k''(t) + \lambda_k T_k(t) = a_k \sin t, \tag{14}$$

$$T_k(0) = b_k, \quad T_k'(0) = -a_k, \tag{15}$$

where $\lambda_k = (k\pi)^2$.

Step 4 (**Solve ODE for $T_k(t)$**). It is also a good exercise to solve the above ODE. If you are very familiar with this type of equations, the solution should have the following general form

$$T_k(t) = c_k \cos(k\pi t) + d_k \sin(k\pi t) + p_k \sin t. \tag{16}$$

We plug (16) into (14) and (15) to find

$$(\pi^2 k^2 - 1)p_k \sin t = a_k \sin t, \tag{17}$$

$$c_k = b_k, \quad (k\pi)d_k + p_k = -a_k. \tag{18}$$

Since $\pi^2 k^2 \neq 1$,

$$p_k = \frac{a_k}{\pi^2 k^2 - 1}, \quad c_k = b_k, \quad d_k = -\frac{\pi k a_k}{\pi^2 k^2 - 1}. \tag{19}$$

If you are not so familiar with this type of ODE, let us solve it step by step. First one has to factorize the differential operator in (14), i.e.

$$\left[\frac{d}{dt} + ik\pi\right]\left[\frac{d}{dt} - ik\pi\right] T_k(t) = a_k \sin t. \tag{20}$$

Then we solve these two first-order ODE one by one as follows.

$$\begin{aligned}
e^{ik\pi t}\left[\frac{d}{dt} + ik\pi\right]\left[\frac{d}{dt} - ik\pi\right] T_k(t) &= a_k \sin t \cdot e^{ik\pi t}, \\
\frac{d}{dt}\left(e^{ik\pi t}\left[\frac{d}{dt} - ik\pi\right] T_k(t)\right) &= \frac{a_k}{2i}(e^{it} - e^{-it}) \cdot e^{ik\pi t}, \\
\frac{d}{dt}\left(e^{ik\pi t}\left[\frac{d}{dt} - ik\pi\right] T_k(t)\right) &= \frac{a_k}{2i}\left(e^{it(k\pi+1)} - e^{it(k\pi-1)}\right).
\end{aligned} \tag{21}$$

Integrate in t to reduce it to a first-order ODE

$$\begin{aligned}
e^{ik\pi t}\left[\frac{d}{dt} - ik\pi\right] T_k(t) &= \frac{a_k}{2i}\left(\frac{e^{it(k\pi+1)}}{i(k\pi+1)} - \frac{e^{it(k\pi-1)}}{i(k\pi-1)}\right) + C_0, \\
e^{-ik\pi t}\left[\frac{d}{dt} - ik\pi\right] T_k(t) &= \frac{a_k}{2i}\left(\frac{e^{it(-k\pi+1)}}{i(k\pi+1)} - \frac{e^{it(-k\pi-1)}}{i(k\pi-1)}\right) + C_0 e^{-2ik\pi t}, \\
\frac{d}{dt}\left[e^{-ik\pi t} T_k(t)\right] &= \frac{a_k}{2i}\left(\frac{e^{it(-k\pi+1)}}{i(k\pi+1)} - \frac{e^{it(-k\pi-1)}}{i(k\pi-1)}\right) + C_0 e^{-2ik\pi t}, \\
\frac{d}{dt}\left[e^{-ik\pi t} T_k(t)\right] &= -\frac{a_k}{2}\left(\frac{e^{it(-k\pi+1)}}{k\pi+1} - \frac{e^{it(-k\pi-1)}}{k\pi-1}\right) + C_0 e^{-2ik\pi t}, \\
e^{-ik\pi t} T_k(t) &= -\frac{a_k}{2}\left(\frac{e^{it(-k\pi+1)}}{i(k\pi+1)(-k\pi+1)} - \frac{e^{it(-k\pi-1)}}{i(k\pi-1)(-k\pi-1)}\right) + \frac{C_0}{-2ik\pi} e^{-2ik\pi t} + D, \\
T_k(t) &= -\frac{a_k}{2}\left(\frac{e^{it}}{i(k\pi+1)(-k\pi+1)} - \frac{e^{-it}}{i(k\pi-1)(-k\pi-1)}\right) + Ce^{-ik\pi t} + De^{ik\pi t}, \\
T_k(t) &= \frac{a_k}{2i}\left(\frac{e^{it}}{k^2\pi^2 - 1} - \frac{e^{-it}}{k^2\pi^2 - 1}\right) + Ce^{-ik\pi t} + De^{ik\pi t}, \\
T_k(t) &= \frac{a_k \sin t}{k^2\pi^2 - 1} + \tilde{C}\cos(k\pi t) + \tilde{D}\sin(k\pi t).
\end{aligned}$$

Here $\tilde{C}$ and $\tilde{D}$ need to be determined by initial conditions (15),

$$\tilde{C} = T_k(0) = b_k, \quad \frac{a_k}{k^2\pi^2 - 1} + k\pi\tilde{D} = T_k'(0) = -a_k. \tag{22}$$

That gives

$$T_k(t) = \frac{a_k \sin t}{k^2\pi^2 - 1} + b_k \cos(k\pi t) - \frac{\pi k a_k}{\pi^2 k^2 - 1}\sin(k\pi t). \tag{23}$$

Note that this agree with (19).

As a final remark to this part, if the right-hand side of (14) *happens to be* e.g. $a_k \sin \pi t$, the general solution (16) for $k = 1$ has to include a new term characterizing *resonance*

$$T_1(t) = c_1 \cos(\pi t) + d_1 \sin(\pi t) + p_1 t \sin(\pi t). \tag{24}$$

Note the last term has a linear growth. For the other k's, the general solutions are not affected. This is also clear in (21); indeed, in this new case, we will have

$$\frac{d}{dt}\left(e^{ik\pi t}\left[\frac{d}{dt} - ik\pi\right] T_k(t)\right) = \frac{a_k}{2i}\left(e^{2itk\pi} - 1\right). \tag{25}$$

in the place of (21). Taking integral will not only give us the usual exponential term, but also a term linear in t.

We are lucky that there is no resonance in this problem since $1 \neq k\pi$; I mention this just in case you would come across such situation somewhere in future.

Step 5 (**Put everything together**). By (8), (23) and $u = w + v$, we can write down the solution to $u(x,t)$

$$u(x,t) = x \sin t + \sum_{k=1}^{\infty} \left[\frac{a_k \sin t}{k^2\pi^2 - 1} + b_k \cos(k\pi t) - \frac{\pi k a_k}{k^2\pi^2 - 1} \sin(k\pi t)\right] \sin(k\pi x), \tag{26}$$

where

$$a_k = \frac{(-1)^{k+1} \cdot 2}{k\pi}, \quad b_k = \frac{4(1 - (-1)^k)}{(k\pi)^3}. \tag{27}$$

The final remark for this problem is that if you really want to simplify the solution a little bit, we observe that

$$x = \sum_{k=1}^{\infty} a_k X_k(x).$$

So we can write the first term in (26) into a sine series and simplify the formula. We will have

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} a_k \sin t \cdot \sin(k\pi x) + \sum_{k=1}^{\infty} \left[\frac{a_k \sin t}{k^2\pi^2 - 1} + b_k \cos(k\pi t) - \frac{\pi k a_k}{k^2\pi^2 - 1} \sin(k\pi t)\right] \sin(k\pi x) \\ &= \sum_{k=1}^{\infty} \left[\frac{k^2\pi^2 a_k \sin t}{k^2\pi^2 - 1} + b_k \cos(k\pi t) - \frac{\pi k a_k}{k^2\pi^2 - 1} \sin(k\pi t)\right] \sin(k\pi x). \end{aligned}$$

□