The Unfinished Problem in the Reciation on Feb. 19, 2016

Jiajun Tong

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Problem 1 (Problem 4.9 on page 54 of EPDE, with modification)**.** *Show that*

$$2u_{xx} + 5u_{xt} + 3u_{tt} = 0 \tag{1}$$

is hypobolic, and solve this equation on $\mathbb{R}$ *with initial conditions*

$$u(x,0) = 0, \quad u_t(x,0) = x\mathrm{e}^{-x^2}. \tag{2}$$

Proof. Recall the canonical form of general second order PDE is that

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = \text{lower order terms like } u_x,\ u_y \text{ etc..}$$

The equation is hyperbolic if discriminant $B^2 - AC > 0$. For (1), we have

$$A = 2, \quad B = \frac{5}{2}, \quad C = 3, \quad \text{and } B^2 - AC = \frac{25}{4} - 6 = \frac{1}{4} > 0.$$

Hence, (1) is hypobolic.

An alternative and equivalent way is to consider the *symbol* of the second order differential operator in (1) (for a quick reference, see *symbol of a differential operator* page on Wikipedia)

$$2\partial_{xx} + 5\partial_{xt} + 3\partial_{tt} \rightsquigarrow 2\xi^2 + 5\xi\eta + 3\eta^2.$$

The equation (1) is hypobolic if and only if we can factorize the symbol with coefficients in $\mathbb{R}$. Indeed,

$$2\xi^2 + 5\xi\eta + 3\eta^2 = (2\xi + 3\eta)(\xi + \eta).$$

As a consequence, (1) is hyperbolic and it becomes

$$(2\partial_{xx} + 5\partial_{xt} + 3\partial_{tt})u = (2\partial_x + 3\partial_t)(\partial_x + \partial_t)u. \tag{3}$$

To solve the equation, we first look for change of variable

$$\eta = ax + bt,$$
$$\xi = cx + dt,$$

together with its inverse

$$x = \tilde{a}\eta + \tilde{b}\xi,$$
$$t = \tilde{c}\eta + \tilde{d}\xi,$$

where

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

We expect that under the new (η, ξ)-coordinate, (1) becomes

$$\partial_{\eta\xi} u = 0. \tag{4}$$

To achieve this, we compute by chain rule,

$$\begin{aligned} \frac{\partial}{\partial \eta} &= \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \eta}\frac{\partial}{\partial t} = \tilde{a}\frac{\partial}{\partial x} + \tilde{c}\frac{\partial}{\partial t}, \\ \frac{\partial}{\partial \xi} &= \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t} = \tilde{b}\frac{\partial}{\partial x} + \tilde{d}\frac{\partial}{\partial t}. \end{aligned}$$

Hence,

$$\partial_{\eta\xi} u = (\tilde{a}\partial_x + \tilde{c}\partial_t)(\tilde{b}\partial_x + \tilde{d}\partial_t)u. \tag{5}$$

Comparing (5) with (3), it is natural to choose

$$\tilde{a} = 2, \quad \tilde{c} = 3, \quad \tilde{b} = 1, \quad \tilde{d} = 1.$$

Therefore,

$$\begin{aligned} x &= 2\eta + \xi, \\ t &= 3\eta + \xi, \end{aligned}$$

and thus

$$\begin{aligned} \eta &= -x + t, \\ \xi &= 3x - 2t. \end{aligned} \tag{6}$$

Now under this change of variable the equation becomes (4). We simply integrate (4) in η and then ξ to find

$$\begin{aligned} \partial_{\eta\xi} u = 0 &\Rightarrow \partial_\xi u = f(\xi), \\ &\Rightarrow u = F(\xi) + G(\eta), \end{aligned}$$

with F and G to be determined. By (6), in (x, t)-coordinate, we have

$$u(x, t) = F(3x - 2t) + G(-x + t). \tag{7}$$

Take t-derivative and we find

$$u_t(x, t) = -2F'(3x - 2t) + G'(-x + t).$$

Let $t = 0$ in the above two equations and use (2),

$$\begin{aligned} 0 &= u(x, 0) = F(3x) + G(-x), \\ x\mathrm{e}^{-x^2} &= u_t(x, 0) = -2F'(3x) + G'(-x). \end{aligned} \tag{8}$$

To solve the above equations, we take x-derivative to the first equation and find

$$\begin{aligned} &3F'(3x) - G'(-x) = 0, \\ &-2F'(3x) + G'(-x) = x\mathrm{e}^{-x^2}. \end{aligned}$$

By solving this linear system in $F'(3x)$ and $G'(-x)$, we find

$$F'(3x) = x\mathrm{e}^{-x^2},$$
$$G'(-x) = 3x\mathrm{e}^{-x^2}.$$

By a change of variable

$$G'(y) = -3y\mathrm{e}^{-y^2}.$$

To find out $G(y)$, we simply integrate in y

$$\begin{aligned} G(y) &= C + \int_0^y -3s\mathrm{e}^{-s^2}\,ds = C - \frac{3}{2}\int_0^y \mathrm{e}^{-s^2}\,ds^2 \\ &= C - \frac{3}{2}\int_0^{y^2} \mathrm{e}^{-z}\,dz = C_0 + \frac{3}{2}\int_0^{y^2} d\mathrm{e}^{-z} \\ &= C + \frac{3}{2}\left(\mathrm{e}^{-y^2} - 1\right) = \frac{3}{2}\mathrm{e}^{-y^2} + C_0, \end{aligned}$$

where in the last equation, the constant in absorbed into the definition of C_0. By the first equation in (8), we know that

$$F(3x) = -G(-x) = -\frac{3}{2}\mathrm{e}^{-x^2} - C_0.$$

Thus

$$F(y) = -\frac{3}{2}\mathrm{e}^{-\left(\frac{y}{3}\right)^2} - C_0 = -\frac{3}{2}\mathrm{e}^{-\frac{y^2}{9}} - C_0.$$

Therefore, by (7),

$$u(x,t) = -\frac{3}{2}\mathrm{e}^{-\frac{(3x-2t)^2}{9}} + \frac{3}{2}\mathrm{e}^{-(-x+t)^2}. \tag{9}$$

Note that the constants C_0 in F and G cancel. One can check that this is indeed the solution.

Another approach of solving (1) is that, with the factorization (3) in hand, we define

$$v(x,t) = (\partial_x + \partial_t)u(x,t).$$

Then we can rewrite (1) as

$$\begin{aligned} &(2\partial_x + 3\partial_t)v(x,t) = 0, \\ &(\partial_x + \partial_t)u(x,t) = v(x,t). \\ &u(x,0) = 0, \quad u_t(x,0) = x\mathrm{e}^{-x^2}. \end{aligned}$$

Now (1) becomes two first order equations. We may solve for v first and then for u. However, we need initial condition for v. To find out that, by definition,

$$v(x,0) = (\partial_x + \partial_t)u(x,0) = \partial_x u(x,0) + \partial_t u(x,0) = 0 + x\mathrm{e}^{-x^2}.$$

The last equation comes from taking x-derivative to the initial condition for u and using the initial condition for u_t, see (2). Hence, (1) becomes

$$(2\partial_x + 3\partial_t)v(x,t) = 0, \quad v(x,0) = x\mathrm{e}^{-x^2}, \tag{10}$$
$$(\partial_x + \partial_t)u(x,t) = v(x,t), \quad u(x,0) = 0. \tag{11}$$

Using the method of characteristics, for (10), we take

$$\frac{dt}{ds} = 3, \quad \frac{dx}{ds} = 2, \quad \frac{dv}{ds} = 0.$$

This implies v is constant on the lines with $3x - 2t =$ constant or $x - \frac{2}{3}t =$ constant. Note that this agrees with ξ in (6) in the previous approach. Therefore, we find the solution for v

$$v(x,t) = \left(x - \frac{2}{3}t\right) \mathrm{e}^{-\left(x - \frac{2}{3}t\right)^2}.$$

To this end, (11) becomes

$$(\partial_x + \partial_t)u(x,t) = \left(x - \frac{2}{3}t\right) \mathrm{e}^{-\left(x - \frac{2}{3}t\right)^2}, \tag{12}$$

$$u(x,0) = 0. \tag{13}$$

We leave it as an exercise. Eventually one should be able to reach (9). □