PDE Spring 2016

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Review notes for midterm

Important PDEs

1. Heat or diffusion eq:

$$u_t = k\, u_{xx} \quad \text{in 1D}, \quad k>0$$

$$u_t = k \nabla^2 u \quad \text{in any dimension}$$

2. Advection equation

$$u_t + c\, u_x = 0 \quad \text{in 1D}$$

$$u_t + \vec{c} \cdot \vec{\nabla} u = 0 \quad \text{in any dimension}$$

3. Laplace and Poisson equation:

$$\nabla^2 u = f \quad (\text{Poisson})$$

$$\nabla^2 u = 0 \quad (\text{Laplace})$$

4. Inviscid or Viscous Burgers

$$u_t + u u_x = k\, u_{xx}$$

IVP = initial value problem
BVP = boundary value problem

1. Advection equation:
Needs initial conditions and boundary condition on boundaries (in space-time domain) where characteristics go into the domain

2. Heat eq:
Needs initial conditions and on all boundaries one of:

Dirichlet BC: $u(0,t) = g(t)$
Neumann BC $u_x(0,t) = h(t)$
Mixed/Robin BC $\alpha(t)u + \beta(t)u_x = f(t)$

In higher dimensions Neumann is

$$\frac{\partial u}{\partial n} = \vec{n} \cdot \vec{\nabla} u = \text{given on boundary}$$

normal vector

3. For Laplace/Poisson same as heat but no initial conditions

A is linear if it has the form:

$$\begin{cases} \mathcal{L}u = f(x,t) \leftarrow \text{PDE} \\ \mathcal{B}u = h(x,t) \leftarrow \text{BCs} \end{cases}$$

Linear operator

If $f = h = 0$ — homogeneous eq.

Superposition principle:
Sums and integrals of solutions are solutions, specifically:

1 If $\mathcal{L}v = 0$ then if $\mathcal{L}u = f$ solves the BVP so does $u + \alpha v$

arbitrary coefficient

2 $\mathcal{L}u = f$ has a unique solution iff $\mathcal{L}v = 0 \Leftrightarrow v = 0$

3 If $u(x,t;\eta)$ is a solution, so is

$$\int c(\eta)\, u(x,t;\eta)\, d\eta$$

A BVP is well-posed if:

$$\left\{\begin{array}{l} \text{— existence} \\ \text{— uniqueness} \\ \text{— stability} \end{array}\right.$$

Stability means that

$$\left\{\begin{array}{l} \mathcal{L}u = f \\ \mathcal{L}(u+\delta u) = f+\delta f \end{array}\right. \Longrightarrow \|\delta u\| \le C\|\delta f\|$$

any norm suitable to the problem

Conservation laws

In $\underline{1D}$:

$$u_t + \varphi_x = f$$

strong form of PDE

flux source

e.g.
$$\left\{\begin{array}{ll} \varphi = uc & \text{for advection} \\ \varphi = -k u_x & \text{for diffusion} \end{array}\right.$$

In *higher dimensions*

$$u_t + \vec{\nabla}\cdot\vec{\varphi} = f$$

PDE classification
for second-order PDEs

$$\mathcal{L}u = a u_{xx} + 2b u_{xy} + c u_{yy} = 0$$

$$d = b^2 - ac$$

The discriminant determines

1. $d > 0$ hyperbolic $\Rightarrow$

$$\mathcal{L} = (\alpha \partial_x + \beta \partial_y)(\gamma \partial_x + \delta \partial_y)$$

$\Rightarrow$ Change coordinates so that

$$\mathcal{L} = \partial_s \, \partial_t = \partial_{st}$$

2. $d = 0$ parabolic

$$\mathcal{L} = (\alpha \partial_x + \beta \partial_y)^2$$

$$\mathcal{L} = \partial_{ss}$$

3. $d < 0$

$\mathcal{L}$ cannot be factored in real factors

$$\mathcal{L} = \partial_{ss} + \partial_{tt}$$

Advection / One-Way Wave Eq.

$$u_t + c u_x = 0 \Rightarrow$$
$$u = f(x - ct)$$

Method of characteristics:

$$p u_x + q u_y = f$$

Track solution along characteristic

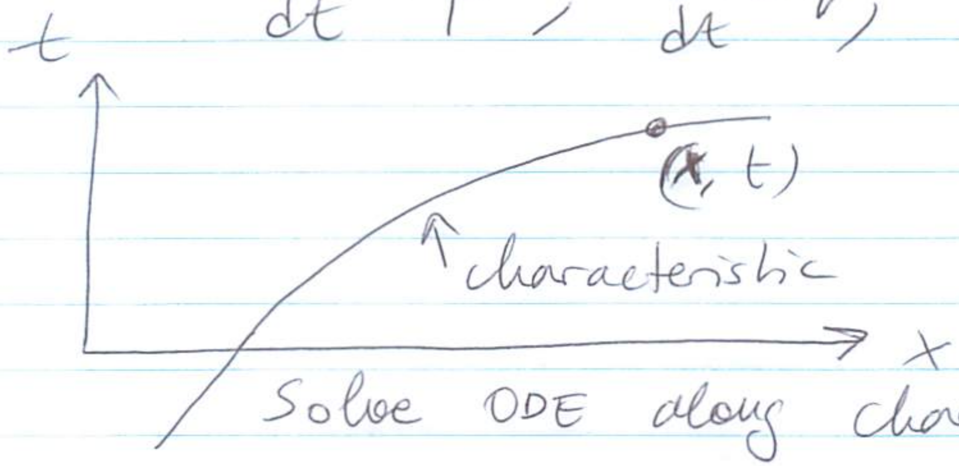
$$w = u(x(t), y(t))$$

$$\frac{dw}{dt} = \underbrace{\frac{dx}{dt}}_{p} u_x + \underbrace{\frac{dy}{dt}}_{q} u_y = f$$

or $$\frac{dx}{p} = \frac{dy}{q} = \frac{dw}{f}$$

versus

$$\frac{dx}{dt} = p, \quad \frac{dy}{dt} = q, \quad \frac{dw}{dt} = f$$



Solve ODE along characteristic

The Wave Equation
or general hyperbolic eqs

$$\mathcal{L} = a u_{xx} + 2b u_{xy} + c u_{yy}$$
$$= (\underbrace{\alpha \partial_x + \beta \partial_y}_{\partial_s})(\underbrace{\gamma \partial_x + \delta \partial_y}_{\partial_t})$$
$$=$$

Find linear transformation

$$\begin{cases} x = \alpha s + \beta t \\ y = \gamma s + \delta t \end{cases}$$

and choose

$$\begin{cases} \partial_s = \alpha \partial_x + \beta \partial_y \\ \partial_t = \gamma \partial_x + \delta \partial_y \end{cases}$$

then solve

$$\partial_{st} u = f(s,t)$$
$$\partial_s(\partial_t u) = f \quad \text{first}$$
$$\partial_t u = \int^s f \, ds' + A(t)$$
$$u = \iint^{t\,s} f \, ds' dt' + B(t) + C(s)$$

Then use initial and / or boundary conditions to try to figure out $B(t)$ and $C(t)$ 8

For wave equation $x \in \mathbb{R}$

$$\begin{cases} u_{tt} = c^2 u_{xx} + f(x,t) \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

D'Alambert's formula
(know how it is derived!)

$$u = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

If $f \neq 0$ $\left\{ + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)\,ds\,d\tau \right.$

We derived this using d'Alambert but you can also derive it directly from change of variables
→ Try it!

Diffusion Equation

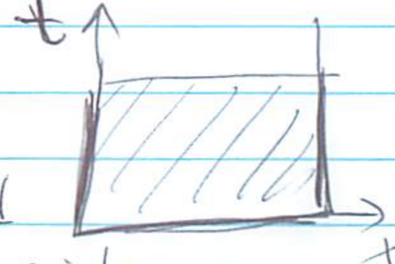
$$\begin{cases} u_t = k u_{xx} & x \in \mathbb{R} \\ u(x,0) = \varphi(x) \end{cases}$$

$$\Rightarrow u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\,\varphi(y)\,dy$$

where G is the Green's function, i.e., the solution of

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = \delta(x) \end{cases} \quad \text{or} \quad \begin{cases} u_t = \delta(x)\delta(t) \\ u(x,-\infty) = 0 \end{cases}$$

$$G = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/(4kt)}$$



$\boxed{\text{Maximum principle}}$

Extremum is achieved on one of the three sides: initial conditions or boundary

"Energy" $E = \frac{1}{2}\int_0^{\ell} u^2\,dx$

$$\frac{dE}{dt} < 0 \text{ if } u_x \neq 0$$

$$= -k\int (u_x)^2\,dx$$

Duhamel's Principle

$$u(x,t) = \mathcal{F}_t(\varphi(\cdot))(x)$$

$$u(x,t) = \mathcal{F}_t \varphi(x)$$

(linear) solution operator or forward map for homogeneous equation

$$\begin{cases} \mathcal{L} u = 0 \\ u(x,0) = \varphi(x) \end{cases}$$

Then the solution of

$$\begin{cases} \mathcal{L} u = f \\ u(x,0) = \varphi(x) \end{cases}$$

is

$$u(x,t) = \mathcal{F}_t(\varphi) + \int_0^t \mathcal{F}_{t-s} f(x,s)\,ds$$

Note: This assumes that the equation is time invariant: looks the same at all times. If PDE depends on time explicitly then $\mathcal{F}_{t,s}$ and not $t-s$