PDE Spring 2016

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Lecture 8

Diffusion Equation

We consider here the heat equation on an unbounded domain in 1D:

$$\begin{cases} u_t = k u_{xx}, \quad k>0 \\ x \in \mathbb{R}, \quad t>0 \\ u(x,0) = \varphi(x) \quad \text{IC} \end{cases}$$

Cauchy problem for diffusion

How do we solve this?

We will take a somewhat convoluted route, but this is required since we do not yet understand distributions...

Let's first solve the hot-cold problem:

$$\begin{cases} w_t = k\, w_{xx} \\ w(x,0) = 0 \quad \text{for } x < 0 \\ w(x,0) = 1 \quad \text{for } x > 0 \end{cases}$$

and then use the solution to solve the general Cauchy problem.

We already discussed dimensional analysis for the heat equation.

k has units $\left[\frac{m^2}{s}\right]$

the solution can only depend on $\sqrt{kt}$ which has units m (length)

The IVP is scale-invariant, i.e. changing the unit of meter gives exactly the same problem and thus the same solution!

So solution can only depend on $x/\sqrt{kt}$

For convenience later on we will add a factor of 4:

$$w = f\left(\frac{x}{\sqrt{4kt}}\right) = f(z)$$

where

$$z = \frac{x}{\sqrt{4kt}} \text{ is } \underline{\text{dimensionless}}$$

Now we know that

$$w = f(z) \Rightarrow$$

$$\begin{cases} w_t = f'(z) z_t \\ w_x = f'(z) z_x \\ w_{xx} = f''(z)(z_x)^2 + f'(z)\underbrace{\cancel{z_{xx}}}_{\text{zero}} \\ \quad\;\; = f''(z)(z_x)^2 \end{cases}$$

After some algebra, putting this back into

$$w_t = k w_{xx}$$

and simplifying (do it!), we get an $\underline{\text{ODE}}$ for $f(z)$!

$$f''(z) + 2z f'(z) = 0$$

Denote $f'(z) = g(z)$

$$g'(z) + 2z\, g(z) = 0$$

$\Rightarrow$ solve this using separation of variables or integrating factor e^{z^2}

$$g(z) = c_1 \exp(-z^2) = f'(z)$$

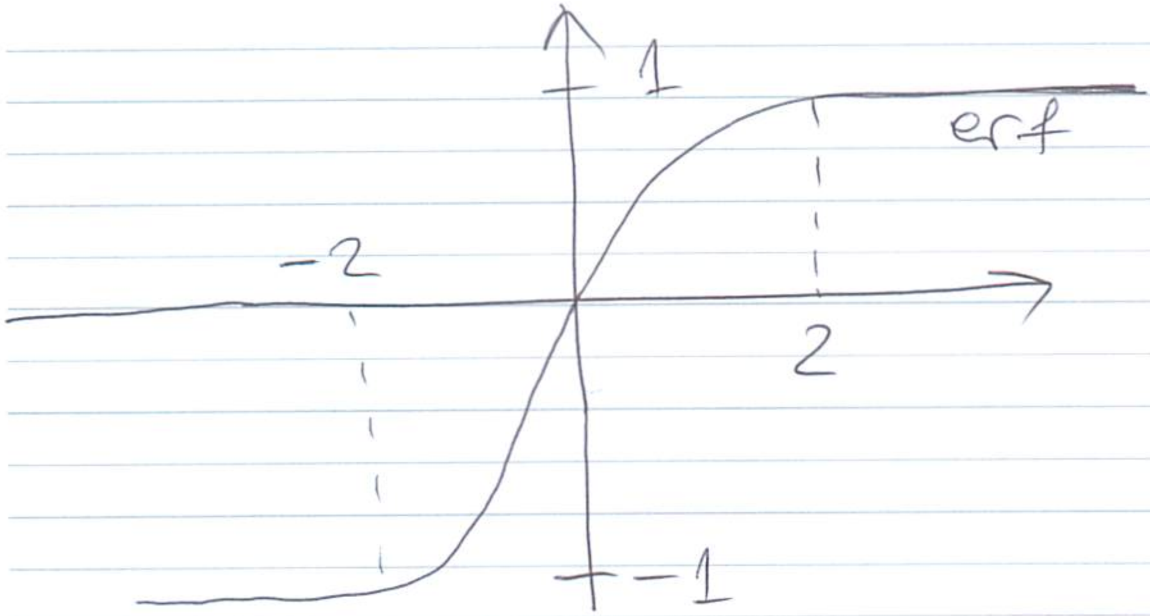
Integrate to get

$$f(z) = c_1 \int_0^z e^{-r^2} dr + c_2$$

The integral does not have a simple form, so let's introduce a new special function

$$\begin{cases} \operatorname{erf}(z) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^z e^{-r^2} dr & \text{ERROR FUNCTION} \\ \operatorname{erfc}(z) = 1 - \operatorname{erf}(z) & \text{COMPLEMEN. ERR. FUNC.} \end{cases}$$

Note $\operatorname{erf}(z) = -\operatorname{erf}(-z)$



For $z \gg 1$, $1 - \mathrm{erf}(z) \simeq \dfrac{e^{-z^2}}{\sqrt{\pi}\, z}$

decays like a Gaussian

So $f(z) = \dfrac{\sqrt{\pi}}{2} c_1 \, \mathrm{erf}(z) + c_2$

Now use ICs but first convert to original coordinates

$$w(x,t) = \frac{c_1\sqrt{\pi}}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right) + c_2$$

$$\lim_{t\to 0} w(x>0,t) = c_1\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(\infty) + c_2$$

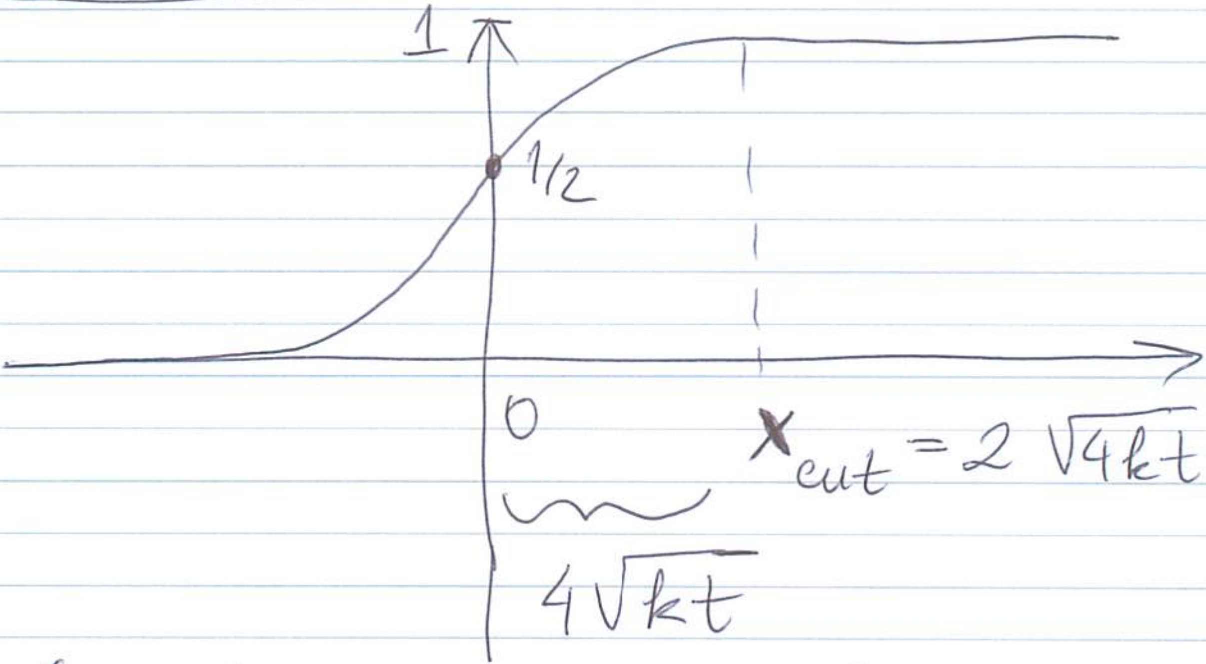
$$= c_1\frac{\sqrt{\pi}}{2} + c_2$$

Similarl for $x<0$.

$$\begin{cases} w(x>0,0) = 1 = c_1 \dfrac{\sqrt{\pi}}{2} + c_2 \\ w(x<0,0) = 0 = -c_1 \dfrac{\sqrt{\pi}}{2} + c_2 \end{cases}$$

$$\Rightarrow \quad c_1 = \frac{1}{\sqrt{\pi}} \qquad c_2 = \frac{1}{2} \quad \Rightarrow$$

$$\boxed{w(x,t) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right]}$$



The distance information propagates by diffusion is on the order of $\sqrt{kt}$, but in principle it reaches everywhere instantaneously (infinetely fast)
[COMPARE ADVECTION]

We now have one solution of a specific IVP for the heat equation. How do we use this to obtain the general solution of

$$u_t = k u_{xx}, \quad u(x,0) = \varphi(x) \;?$$

Observe that:

[Any derivative of a solution is a solution

I.E. if u is a solution so are u_x, u_t, u_{xx}, u_{xt}, etc.

So w_x is also a solution!

$$G(x,t) = w_x(x,t)$$

Proof: $G_t - k G_{xx} =$

$$= w_{xt} - k w_{xxx} =$$

$$= (\underbrace{w_t - k w_{xx}}_{\text{zero}})_x = 0$$

$$\boxed{G(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/(4kt)}}$$

Fundamental solution of heat equation or <u>Green's function</u> or <u>heat kernel</u>

Observe:

$$\int_{-\infty}^{\infty} G(x,t)\,dx = 1 \quad , \quad t>0$$

probability distribution

<u>Claim</u>

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\,\varphi(y)\,dy} \quad (*)$$

is the solution of the IVP

$$u_t = k u_{xx} \quad , \quad u(x,0) = \varphi(x)$$

We already proved in earlier lecture (and homework) that (*) solves the heat equation.

The trickier part is to show that

$$\lim_{t \to 0} u(x,t) = \varphi(x)$$

Recall that $G(x) \equiv \frac{\partial w}{\partial x}$

where $w(x>0,0) = 1$, $w(x<0,0) = 0$

$$u(x,t) = \int G(x-y,t)\,\varphi(y)\,dy =$$

$$= \int \frac{\partial w}{\partial x}(x-y,t)\,\varphi(y)\,dy$$

$$= \int -\frac{\partial}{\partial y}\underbrace{\left(w(x-y,t)\right)\varphi(y)\,dy}_{\text{integrate by parts}}$$

$$= \int w(x-y,t)\,\varphi'(y)\,dy$$

$$\underbrace{-\,w(x-y,t)\,\varphi(y)\Big|_{y=-\infty}^{+\infty}}$$

. Assume boundary term vanishes

$$\lim_{|x|=\infty} \varphi(x) = 0$$

More precisely, we are assuming here that the solution decays to zero at infinity

$$\Rightarrow u(x,t) = \int_{-\infty}^{\infty} w(x-y,t)\,\varphi'(y)\,dy$$

$$\Rightarrow u(x,t=0) = \int_{-\infty}^{\infty} w(x-y,0)\,\varphi'(y)\,dy$$

Recall w only nonzero (one) for $x-y>0$ or $y<x$

$$= \int_{-\infty}^{x} \varphi'(y)\,dy = \varphi(x)$$

since $\varphi(-\infty)=0$

We have thus shown that

$$\lim_{t\to 0} u(x,t) = \varphi(x)$$

as required. Finally, we are able to show that

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \varphi(y)\,dy}$$

is a solution of the IVP we started with.
Note that we have not shown this is a unique solution or that the equation is well-posed yet.

Comment: If $\varphi(x)$ is piecewise continuous, then at a discontinuity

$$\lim_{t\to 0^+} u(x,t) = \frac{1}{2}\left(\varphi(x^-)+\varphi(x^+)\right)$$

Example

Solve the heat equation for
$$u(x,0) = e^{-x}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt} - y}\,dy$$

Complete the square

Change variables

$$p = (y + 2kt - x) / \sqrt{4kt}$$

$$\Rightarrow u(x,t) = e^{kt-x} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-p^2}}{\sqrt{\pi}} dp}_{\text{unity}} = e^{kt-x}$$

This solution grows in time because our initial condition is unphysical and does not decay with ~~xxxx~~. $x \to -\infty$.

In homework you will consider

$$\varphi(x) = \begin{cases} \exp(-x) & \text{for } x > 0 \text{ only} \\ 0 & \text{for } x \le 0 \end{cases}$$

Next class we will show uniqueness and stability and later give a more intuitive derivation of the solution derived here (distributions, delta function, etc)