

PDE Spring 2016

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PDE Classification

$$u \equiv u(x, y)$$

Consider the general second-order linear PDE

$$a u_{xx} + 2b u_{xy} + c u_{yy} + p u_x + q u_y + r u = f$$

where

$$a \equiv a(x, y) \text{ etc.}$$

For PDEs the nature of the equation is (typically) dominated by the highest derivative, so let's delete the first-order terms

$$\mathcal{L}u = f(x, y) \quad (2)$$

$$\mathcal{L} = a \partial_{xx} + 2b \partial_x \partial_y + c \partial_{yy}$$

The nature of the PDE is determined by the

discriminant $d = b^2 - ac$

① $b^2 - ac > 0$ - equation is hyperbolic

This means that with a simple change of coordinates it can be transformed into the equation

$$\mathcal{L}'u(s, t) = g$$

where

$$\mathcal{L}' = \partial_{st}$$

has only a mixed second derivative

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② $b^2 - ac = 0, a > 0$

Eq. is parabolic

meaning that it can be transformed into the equation

$$\mathcal{L}' u(s,t) = g$$

$$\mathcal{L}' = \partial_{ss}$$

so there is only a single second order derivative with no mixed derivatives

③ $b^2 - ac < 0$ - elliptic

$$\mathcal{L}' = \partial_{ss} + \partial_{tt}$$

so there is now a Laplacian appearing

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The canonical form is a form into which every second-order linear equation can be transformed into by a change of independent variables (i.e., of coordinates):

(1) Canonical hyperbolic equation

$$u_{st} + G(s, t, u, u_s, u_t) = 0$$

Example: the wave equation $u_{tt} = \nabla^2 u$

(2) Canonical parabolic equation

$$u_{ss} + H(s, t, u, u_s, u_t) = 0$$

Ex: the heat equation $u_t = \nabla^2 u$

(3) Canonical elliptic eq.

$$u_{ss} + u_{tt} + F(s, t, u, u_s, u_t) = 0$$

Ex: Laplace equation $\nabla^2 u = 0$

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We need to study and understand each type of equation separately, but it is OK to choose one sample prototype equation from each category:

$$\left\{ \begin{array}{l} \text{wave eq. } u_{tt} = \nabla^2 u - \text{hyperbolic} \\ \text{heat eq. } u_t = \nabla^2 u - \text{parabolic} \\ \text{Laplace eq. } \nabla^2 u = 0 - \text{elliptic} \end{array} \right.$$

One way to think of this is as trying to factorize the operator

$$\mathcal{L} = a \partial_x^2 + 2b \partial_{xy} + c \partial_y^2$$

just like one would factorize a quadratic polynomial:

there are either two distinct real roots, one degenerate real root, or two complex-conjugate roots

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We would write

$$a^2 - b^2 = (a-b)(a+b)$$

so

$$\mathcal{L}u = u_{tt} - u_{xx} \quad (\text{wave eq.})$$

$$\mathcal{L} = \partial_{tt} - \partial_{xx} = \partial_t^2 - \partial_x^2 =$$

$$(\partial_t - \partial_x)(\partial_t + \partial_x) = \mathcal{L}^- \mathcal{L}^+$$

$$\begin{cases} \mathcal{L}^- = \partial_t - \partial_x \\ \mathcal{L}^+ = \partial_t + \partial_x \end{cases}$$

What this means is that

$$\mathcal{L}u = \mathcal{L}^-(\mathcal{L}^+u) = \mathcal{L}^+(\mathcal{L}^-u)$$

Exercise: Show this explicitly

Observe \mathcal{L}^- and \mathcal{L}^+ are associated with the advection equations with speed ± 1 .
The advection equation is a degenerate kind of hyperbolic equation

Hyperbolic : (wave-like) (7)

$$\mathcal{L} = (\alpha \partial_x + \beta \partial_y)(\gamma \partial_x + \delta \partial_y)$$

with $\frac{\beta}{\alpha} \neq \frac{\delta}{\gamma}$

and $\alpha, \beta, \gamma, \delta$ are real

Parabolic : (diffusive)

$$\mathcal{L} = (\alpha \partial_x + \beta \partial_y)^2$$

Elliptic (static)

$$\mathcal{L} = (\alpha \partial_x + i\beta \partial_y)(\alpha \partial_x - i\beta \partial_y)$$

not most general

$$= \alpha^2 \partial_x^2 + \beta^2 \partial_y^2$$

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Higher-order PDEs

$$u_{ttt} + u_{t_{xx}} + u u_{ttx} + u u_{xxx} = 0 =$$

$$(\partial_t + u \partial_x) (u_{tt} + u_{xx})$$

$$= \underbrace{(\partial_t + u \partial_x)}_{\text{hyperbolic piece}} \underbrace{(\partial_t + i \partial_x)(\partial_t - i \partial_x)}_{\text{elliptic piece}} u$$

So this is a mixed hyperbolic-elliptic equation

But this classification is not as useful anymore...