PDE Spring 2016

Lecture 4

Conservation Laws

Reading: EPDE: 3.2, 3.2.1
APDE: Section 1.2, Ex. 1.15
1.3, 1.7

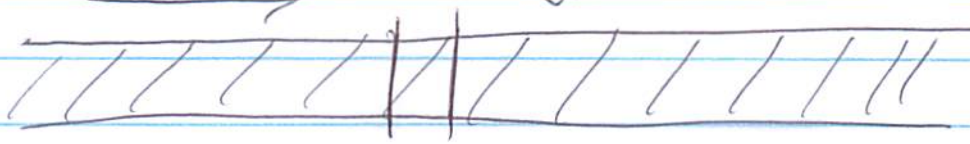
We consider the transport of a conserved quantity in 1D

Examples:

→ Traffic flow

→ Polutant / smoke in air flow

→ Heat (energy) flow through metal



Traffic Lane on Highway

$u(x,t)$ = density of cars / molecules

Quantity = $u(x,t)\,dx$

Flux $\varphi(x,t)$ = amount of quantity passing through x

Positive means to the right (2)

E.g. number of cars passing through intersection / mile marker per second

$f(x,t)$ = source / sink of quantity

E.g: Traffic coming onto highway from entrance (+) or leaving from exit (−)

Fundamental conservation law

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \varphi(a,t) - \varphi(b,t) + \int_a^b f(x,t)\,dx$$

This is called a "weak form" of a conservation PDE. Under some assumptions, it can be converted into a traditional PDE, which is called the "strong form" and is the focus of this course.

Recall from ODEs that

$$x'(t) = f(x,t) \text{ is equivalent to}$$

integral equation

$$x(t) = x(0) + \int_0^t f(x(t),t)\,dt$$

which is in fact more general than ODE (e.g. stochastic ODEs)

If function is sufficiently smooth, i.e. sufficiently continuously differentiable, we can convert conservation law to a PDE.

Lots of PDEs in practice come from conservation laws!

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \int_a^b u_t\,dx$$

<u>if</u> u has continuous first partial derivatives

Similarly, if φ has continuous first partials, then fund. th. of calculus says

$$\varphi(a,t) - \varphi(b,t) = -\int_a^b \varphi_x\,dx$$

Conservation law becomes:

$$\int_a^b \left[u_t + \varphi_x - f \right] dx = 0$$

for all a and b

Since integrand is continuous (crucial!) and a and b are arbitrary, the above implies integrand identically vanishes

$$\boxed{u_t + \varphi_x = f}$$

conservation law PDE

To turn this into a PDE we need a <u>constitutive law</u>

$$\varphi \equiv \varphi(u, x, t)$$

which is problem-specific.

For traffic flow

$$\varphi = u\,c$$

where c is speed of cars

If c is constant (boring!) then we get advection equation

$$u_t + c\,u_x = f$$

But in the real world c depends on u (and also x, t due to road/weather conditions)

So to get an actual PDE we need to do

<u>mathematical modeling</u>

E.g.

$$c = c_{max}\left(1 - \frac{u}{u_{max}}\right)$$

where u_{max} is the "jamming" density of cars

$\Rightarrow$ (derive as practice)

$$u_t + \underbrace{c_{max}\left(1 - 2\frac{u}{u_{max}}\right)}_{\substack{\text{Speed of traffic "waves"} \\ \text{or speed of propagation} \\ \text{of information through} \\ \text{the highway}}} u_x = 0$$

The above is a nonlinear advection equation similar to Burgers equation

We will not study nonlinear advection in detail but see text books for advanced students.

Here, as practice, let's show that the *implicit* solution

$$\begin{cases} u = F(\underbrace{x - c(u)t}_{y}) \\ \text{solves} \\ u_t + c(u)u_x = 0 \end{cases}$$

$$u_t = F'(y)\left[-c(u) - c'(u)u_t t\right]$$

$$u_x = F'(y)\left[1 - c'(u)u_x t\right]$$

$$c u_x = F'(y)\left[c - c'c u_x t\right]$$

$$u_t + c u_x = -c'F'(y)t\left[u_t + c u_x\right]$$

$$\Rightarrow \text{ if } c'F' \neq 0 \quad u_t + c u_x = 0$$

Diffusion
as a conservation law

We all know that heat "flows" from hot to cold.

In general, the higher the gradient of temperature the larger the flow.

So it seems reasonable to postulate

$$\varphi = -k u_x$$

where $k > 0$ is a diffusion constant

$\to$ flux is "down the gradient"

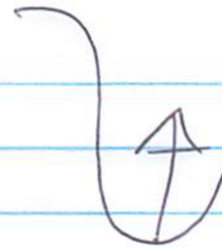
$$\boxed{u_t + (k u_x)_x = 0}$$

Heat equation as a conservation law

Here $k = k(x)$ or even $k = k(u, x, t) > 0$ works

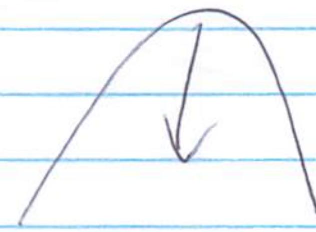
If $u_{xx} > 0$

$\Rightarrow u_t > 0$

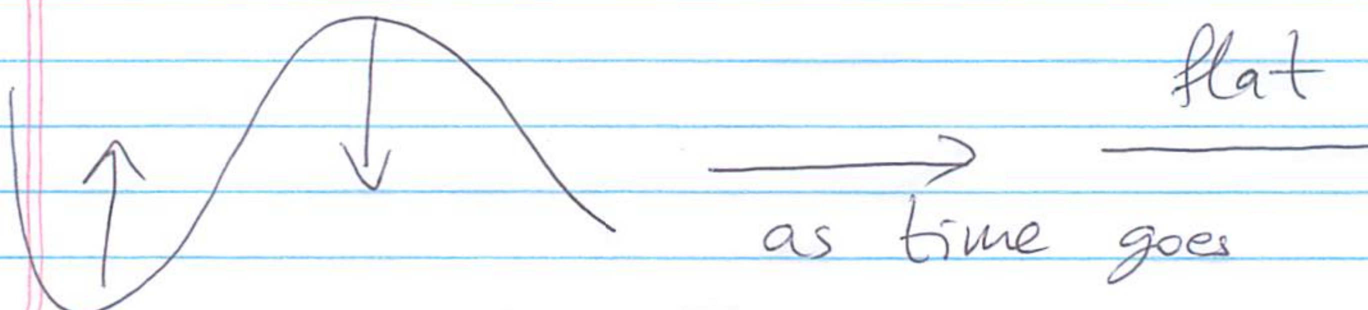


convex

If $u_{xx} < 0$



concave

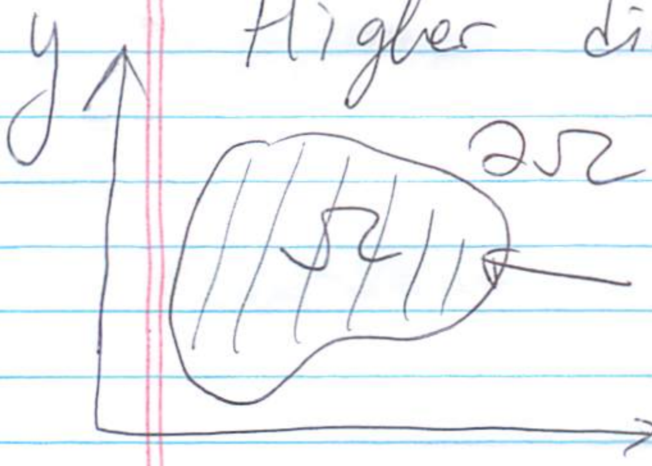


Diffusion "flattens" or "smears" the solution

It makes it smoother with time

There is a deep connection between diffusion and randomness or random walks, see 1.4 in Applied PDE book if interested in details

Higher dimensions



$u(x,y) : (x,y) \in \Omega$

$\partial\Omega$ = boundary of Ω

$$\frac{d}{dt}\int_V u \, dV = \int_{\partial V} (\vec{\varphi} \cdot \vec{n}) \, dS + \int_V f \, dV$$

where $V \subseteq \Omega$ is an arbitrary subvolume

Here flux $\vec{\varphi}$ is a vector and $\vec{\varphi} \cdot \vec{n}$ is the <u>normal</u> flux through the boundary of a subvolume

Since V is fixed with time

$$\frac{d}{dt}\int_V u \, dV = \int_V \frac{\partial u}{\partial t} \, dV$$

Also recall divergence theorem

Green & Gauss generalize fund. theorem of calc.

$$\int_\Omega \nabla \varphi \, dV = \int_{\partial\Omega} \varphi \vec{n} \, ds$$

$$\int_\Omega (\vec{\nabla} \cdot \vec{\varphi}) \, dV = \int_{\partial\Omega} \vec{\varphi} \cdot \vec{n} \, ds$$

$$\Rightarrow \int_V \left(\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{\varphi} - f \right) dV = 0$$

for all $V \subseteq \Omega$

(weak form)

$\Downarrow$ certain assumptions

$$\boxed{\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{\varphi} = f}$$ Strong form

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Advection in higher dimensions $\quad c \to \vec{c}$ (vector)

$\vec{c}(x,y) \to$ vector field in general

$$\vec{\varphi} = u\vec{c}$$

$$\Rightarrow \quad u_t + \vec{\nabla}\cdot(u\vec{c}) = 0$$

if $\vec{c}$ is a constant vector

$$u + \vec{c}\cdot(\vec{\nabla}u) = 0$$

Diffusion in higher dimensions

$$\vec{\varphi} = -k\vec{\nabla}u \quad , \quad k>0$$

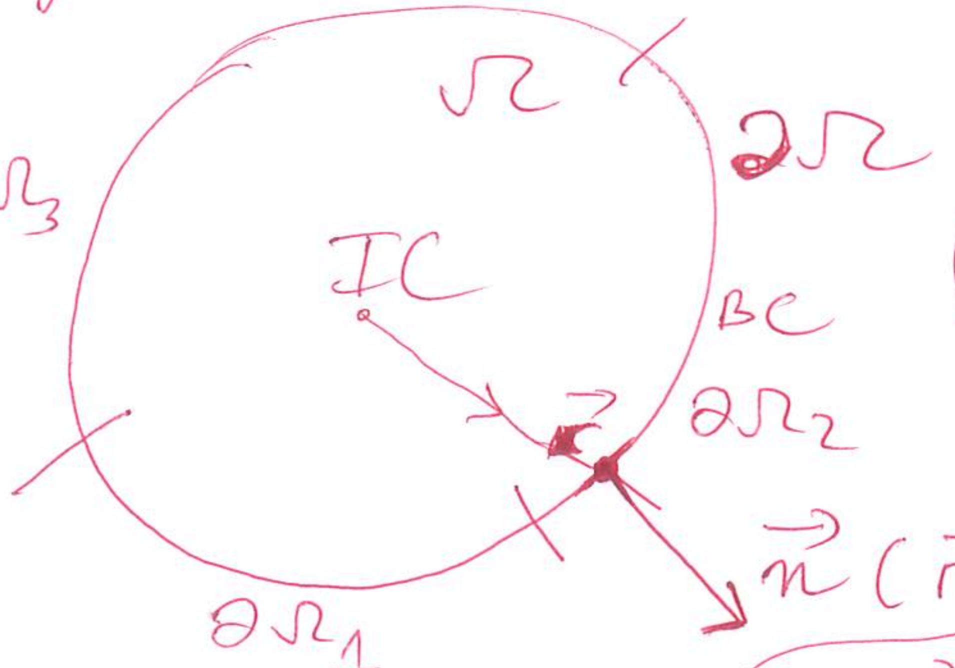
$$u_t + \vec{\nabla}\cdot(-k\vec{\nabla}u) = 0$$

If k constant

$$u_t = k\vec{\nabla}\cdot(\vec{\nabla}u) = k\nabla^2 u$$

Practice: Show $\vec{\nabla}\cdot\vec{\nabla} \equiv \nabla^2$

Higher dimension (heat eq)



Heat eq.

$$\boxed{u_t = \nabla^2 u}$$

IC

$$\boxed{u(\vec{r} \in \Omega) = u_0(\vec{r})}$$

$\partial\Omega_1$: Dirichlet

$$u(x,y) = u(\vec{x}) =$$

$$u(\vec{r}) = f_1(\vec{r}) \; : \forall \vec{r} \in \partial\Omega_1$$

$\partial\Omega_2$: Neumann (Flux BC)

$$\frac{\partial u}{\partial n}(\vec{r}) = (\vec{\nabla} u) \cdot \vec{n}(\vec{r}) = g_2(\vec{r})$$

$$: \forall \vec{r} \in \partial\Omega_2$$

$$\vec{\nabla} u = \begin{pmatrix} \partial u / \partial x \\ \partial u / \partial y \end{pmatrix}$$

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y$$

Ω_3: Robin $\forall \vec{r} \in \partial\Omega_3$

$$\alpha(\vec{r},t)\, u(\vec{r}) + \beta(\vec{r},t) \frac{\partial u}{\partial n}(\vec{r}) = g(\vec{r},t)$$

The Laplace equation

$$\nabla^2 u = 0$$

is the <u>steady state</u> $(t \to \infty)$
limit of the heat eq.
$\to$ No IC any more (no time)
but the <u>same</u> BC's apply

Same goes for Poisson $\nabla^2 u = f(\vec{r})$

How many BCs do we need?
Depends on equation in nontrivial ways

$$u_t + c\, u_x = 0 \qquad c > 0$$