Lecture 3 PDE Classification

Linear differential equations

Operator : Infinite-dimensional generalization of functions, or "function of a function"

$$\mathcal{L}(u) \equiv \mathcal{L}u = +k\,u_{xx}$$

caligraphic letter

∇, $\nabla\cdot$, $\underbrace{\nabla^2 \text{ or } \Delta}_{\text{Laplacian}}$

grad, div

$$u_t = \mathcal{L}u \leftarrow \text{heat equation}$$

or $$\mathcal{L}u = u_t - k\,u_{xx}$$

$$\mathcal{L} = \partial_t - k\,\partial_{xx}$$

$\mathcal{L}u = 0$ Heat eq. in operator notation

Boundary condition

$$\mathcal{B}u = \begin{cases} u(0,t) & \forall t>0,\ x=0 \\ u_x(1,t) & \forall t>0,\ x=1 \end{cases}$$

B.C. $\mathcal{B}u = f(x,t)$

$$\begin{cases} f(0,t) = f_L(t) \\ f(1,t) = h_R(t) \end{cases}$$

$\mathcal{L}$ is a <u>linear operator</u> iff

$$\begin{cases} \mathcal{L}(\alpha u) = \alpha \mathcal{L}(u) \\ \mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v) \end{cases}$$

Check $\mathcal{L} \equiv \partial_{xx}$ $\rightarrow$ follows from properties

Check $\mathcal{L}u = u u_x$

$$\mathcal{L}(\alpha u) = (\alpha u)(\alpha u_x) = \alpha^2 \mathcal{L}(u)$$

<u>NOT</u> linear

Standard Dirichlet, Neumann & Robin BCs are linear

A PDE is linear if it has the form

$$\begin{cases} \mathcal{L}u = f(x,t) \\ \mathcal{B}u = h(x,t) \end{cases}$$

If $f(x,t) \equiv 0$ then the PDE is <u>homogeneous</u> (otherwise non-homogeneous)

All boundary <u>and</u> initial data must be zero.

Superposition principle

1. If u_1 and u_2 are two solutions of a linear PDE then so is any linear combination of them

$$u = \alpha u_1 + \beta u_2 \quad \text{since}$$

$$\mathcal{L}u = \alpha \mathcal{L}u_1 + \beta \mathcal{L}u_2 = 0$$

2. If $\bar{u}$ is a "particular" solution of $\mathcal{L}u = f$ (non homogeneous) then if $\mathcal{L}v = 0$ we have

$$\mathcal{L}(u + \alpha v) = f \quad \text{is another solution}$$

$\Downarrow$ consequence

3. $\mathcal{L}u = f$ has a unique solution iff $v = 0$ is the only solution of $\mathcal{L}v = 0$

Proof: Suppose u_1 / u_2 are sols.

$$\mathcal{L}u_1 = f, \ \mathcal{L}u_2 = f \Rightarrow \mathcal{L}(u_1 - u_2) = 0 \Rightarrow$$

$$u_1 = u_2$$

Linear BVPs have zero, one or infinitely many solutions (just like linear systems $Ax = b$) 5

Well-posedeness

A BVP which has a unique solution that varies continuously with the initial and boundary data is well-posed, otherwise ill-posed

$$\mathcal{L}u = f$$

$$\mathcal{L}(u + \delta u) = f + \delta f$$

δf small $\Rightarrow$ δu small if well posed

$$\|\delta u\| \leq C \|\delta f\|$$

$u_t = u_{xx}$ - well-posed

But $u_t = -u_{xx}$ (backward heat) not well posed

$$u(x,0) = \alpha \cos(nx) \Rightarrow u(x,t) = \alpha e^{n^2 t} \cos(nx)$$

n large $\Rightarrow$ large growth

Continuum superposition

$u(x,t;\eta)$ a solution

η: parameters

$$\Rightarrow v = \int_{\eta \in \mathcal{I}} c(\eta)\, u(x,t;\eta)\, d\eta$$ is also a solution under certain conditions on $c(\eta)$

$$\mathcal{L} v = \mathcal{L} \int_{\eta} c(\eta)\, u(x,t;\eta)\, d\eta =$$

under certain conditions (smoothness) of the integrand we can pass $\mathcal{L}$ inside the integral

$$= \int c(\eta)\, (\mathcal{L} u)\, d\eta = 0$$

If $w = u + i v$ is a complex solution then u and v are real solutions

Classification of PDEs

Linear vs. Non linear
(also semi-linear &
quasi-linear)

But then comes physics!

wave - like (hyperbolic)
diffusion - like (parabolic)
equilibrium (steady-state) (elliptic)

Wave equation classification: 8

$$u(x,t) = Ae^{i(kx-\omega t)}$$

is called a plane wave solution

k = wave number $= \frac{2\pi}{\lambda}$ ← wave length

ω = angular frequency $= \frac{2\pi}{T}$ ← period $= 2\pi f$ ← frequency

$$u_t = D u_{xx}$$

$$-i\omega \cancel{u} = -Dk^2 \cancel{u} \Rightarrow \omega = -iDk^2$$

$\omega = \omega(k)$ ← dispersion relation

If $\omega(k)$ is complex $\Rightarrow$ diffusive

$$u_t + u_{xxx} = 0 \Rightarrow \cancel{+i}\omega \cancel{u} = \cancel{+i}k^3 \cancel{u}$$

$$\omega = k^3$$

$\omega(k)$ is real $\Rightarrow$ wave-like

$\omega''(k) \neq 0 \rightarrow$ dispersive wave eq.

Practice:
$$\begin{cases} u_t + c u_x = 0 \\ u_{tt} = c^2 u_{xx} \end{cases}$$

Change of variables

$$u_t = u_x^2 + u_{xx}$$

$$w = e^u$$

First way:

$$w_t = (e^u) u_t = w u_t =$$

$$= w(u_x^2 + u_{xx})$$

$$w_x = w u_x$$

$$w_{xx} = w_x u_x + w u_{xx} =$$

$$= w u_x^2 + w u_{xx} = w(u_x^2 + u_{xx})$$

$$\Rightarrow w_t = w_{xx} \rightarrow \text{heat eq.}$$

Second way

$u = \ln w$ where $w > 0$

$$u_t = \frac{1}{w} w_t$$

$$u_x = \frac{1}{w} w_x$$

$$u_{xx} = \frac{w_{xx}}{w} - \frac{1}{w^2} w_x^2$$

$$\frac{1}{w} w_t = \cancel{\frac{1}{w^2} w_x^2} + \frac{w_{xx}}{w} - \cancel{\frac{1}{w^2} w_x^2}$$

$$\Rightarrow w_t = w_{xx} \quad \text{if} \quad w \neq 0$$