Lecture 2 PDE Spring 2016

Initial & Boundary conditions

Consider

$$u_{tx} - 4xt = 0$$

$$\partial_t \left(u_x - 2xt^2 \right) = 0 \Rightarrow$$

$$u_x - 2xt^2 = f(x)$$

arbitrary

So the PDE is like a system of infinitely many ODEs (uncountable), one for each x. So we need an initial condition for each x for this ODE.

$$u_x = f(x) + 2xt^2$$

$$u = \int \left[f(x)dx + 2xt^2 \right] dx$$

$u = g(x) + x^2 t^2 + h(t)$

Check

$u_{tx} = 4xt$ and $g(x)$ and $h(t)$ disappear

So to make the solution <u>unique</u>

we need to fix two functions

$g(x)$ and $h(t)$

To fix $g(x)$, we give an <u>initial condition</u> (just like for ODEs)

$$\underbrace{u(x, t=0)}_{IC} = g(x) + h(0) = u_1(x)$$

and to fix $h(t)$ we need

a <u>boundary condition</u>, <u>e.g.</u>

$$u(0, t) = g(0) + h(t) = u_2(t)$$

~~To make the function continuous at the origin we require that~~

$$\cancel{u(0,0) = g(0) + h(0)}$$

Another way to write general solution

$$u(0,0) = g(0) + h(0)$$

$$u(x,t) = u(x,0) + u(0,t) + x^2 t^2 - (h(0) + g(0))$$

$$\Rightarrow \boxed{u(x,t) = x^2 t^2 + u(x,0) + u(0,t) - u(0,0)}$$

Note that for continuity at 0 we require that $u_1(0) = u_2(0) = u(0,0)$

Now consider the advection equation. $u_t + c u_x = 0$

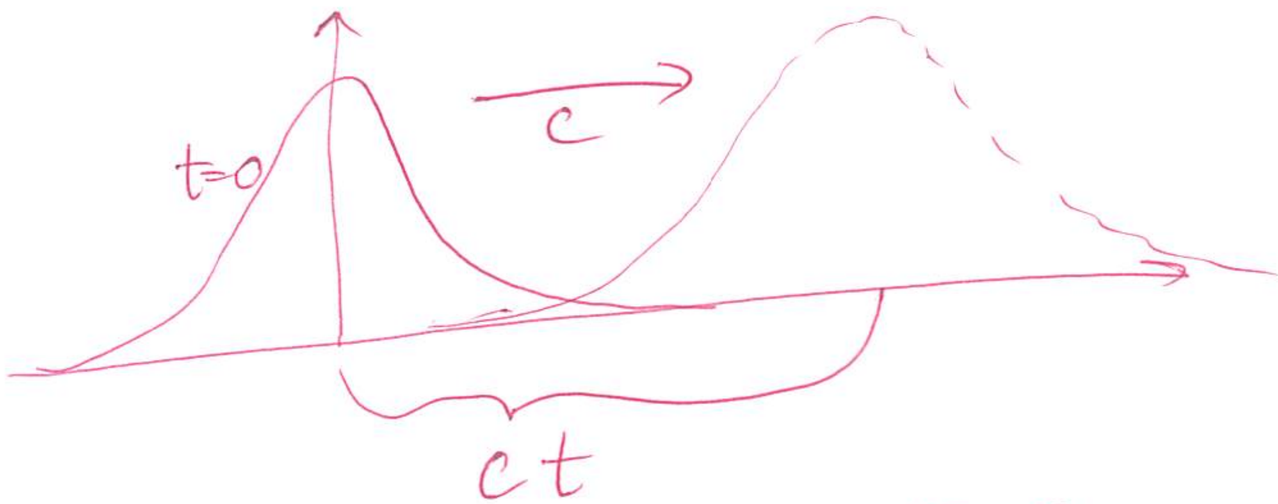
Show that $u = f(x-ct)$ is a solution

$u_t = -cf'(x-ct)$

$c u_x = cf'(x-ct)$

$\Rightarrow u_t + c u_x = 0$

$f(x,0) \equiv u(x,0)$



c has units of $\left[\frac{m}{s}\right]$ = velocity

= speed of propagation.

$$u = f(x-ct)$$

$$u(x, t=0) = f(x) = \text{initial condition}$$

So here we only need an IC $\Rightarrow$ Initial value problem (IVP)
If we also specify BCs we call it a BVP

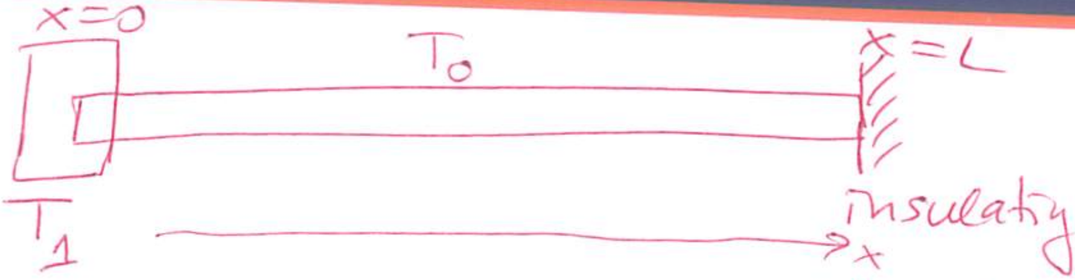
The naming initial versus boundary is arbitrary – in reality these are all boundary conditions in space-time domain.

Wave Equation (IVP)

$$u_{tt} = c^2 u_{xx}$$

Now we need two ICs like ODEs

$$\begin{cases} u(x, t=0) = f(x) \\ u_t(x, t=0) = g(x) \end{cases}$$



(8)

$u \equiv T$ or energy density

$u_t = k\, u_{xx}$

k ← thermal diffusivity

$u(x,0) = T_0$ IC

$u(x=0,t) = T_1$ ← Dirichlet BC

$\partial_x u(x=L,t) = 0$ ← Neumann BC

Here we need two BCs because of the second derivative u_{xx}

Note that we do not require that $T_1 = T_0$, i.e., the IC & the BC are not consistent with each other at $(0,0)$

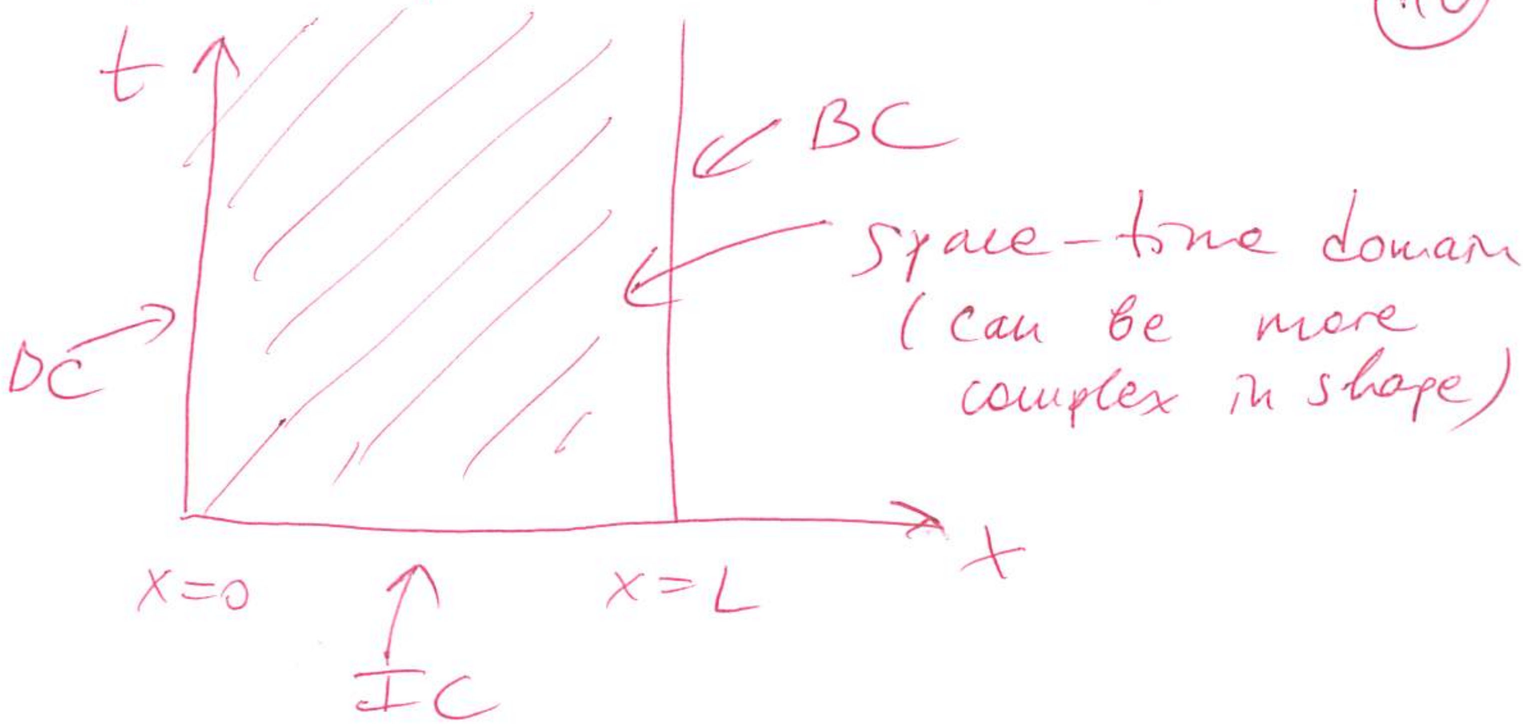
→ The fact this is ok depends on the PDE (functional analysis) the general rule of thumb: 9

The BCs contain lower-order derivatives than the order of the PDE

setting $u_{xx}|_{x=0} = 0$ would not be consistent and there would be no solution

To solve a PDE means to find a function $u(x,t)$: ~~[illegible]~~ $\{x \in I, t > 0\}$ that is sufficiently
open or closed
differentiable and satisfies the PDE and the ICs and BCs

More general heat equation (10)



$$u_t = u_{xx}$$

$$\begin{cases} u(x,0) = f(x) \\ u(0,t) = g_L(t) \\ u(L,t) = g_R(t) \end{cases}$$

$\Big\}$ Dirichlet

or

$$u_x(0,t) = h_L(t)$$ $\}$ Neumann

or $\alpha(t)\, u(0,t) + \beta(t)\, u_x(0,t) = g(t)$

Robin or reflecting BC

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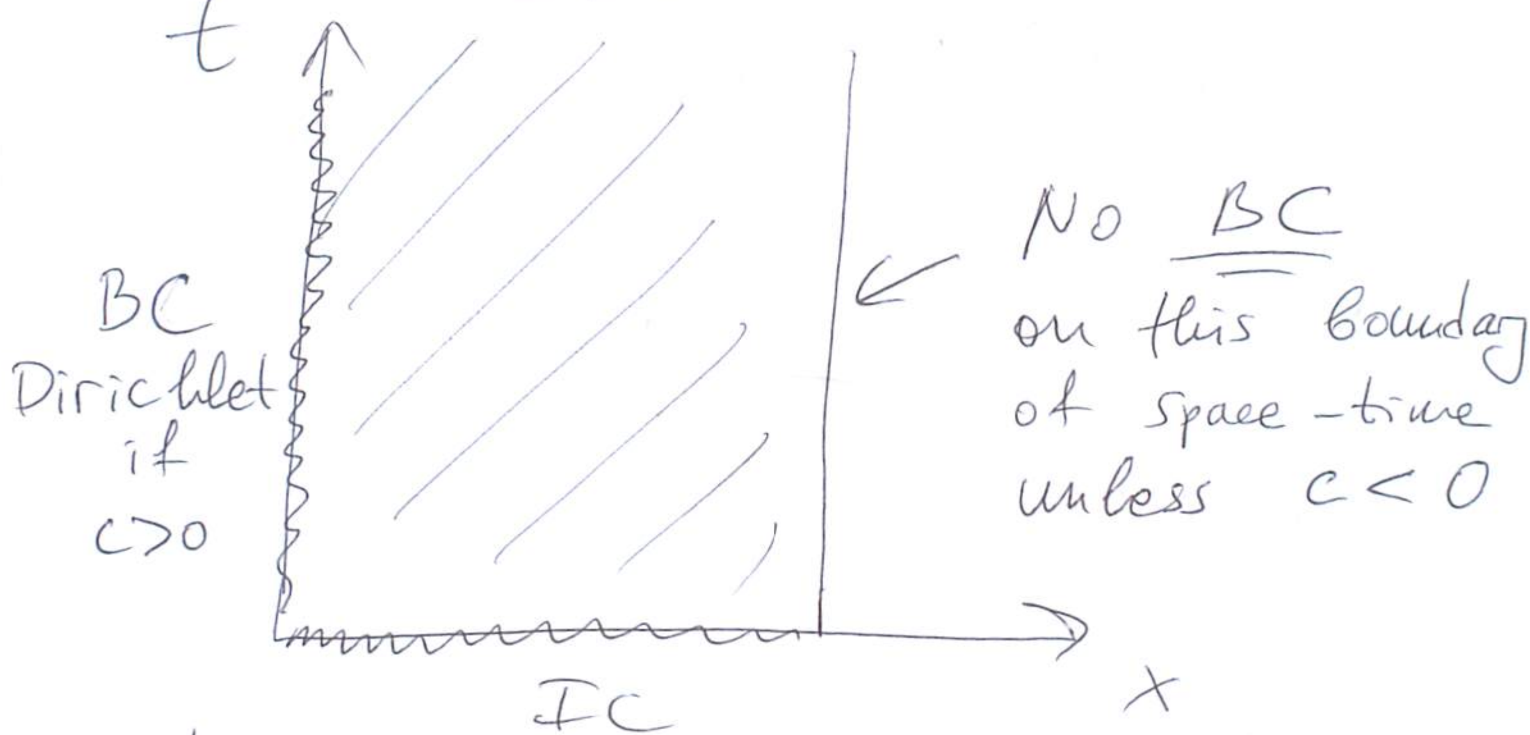
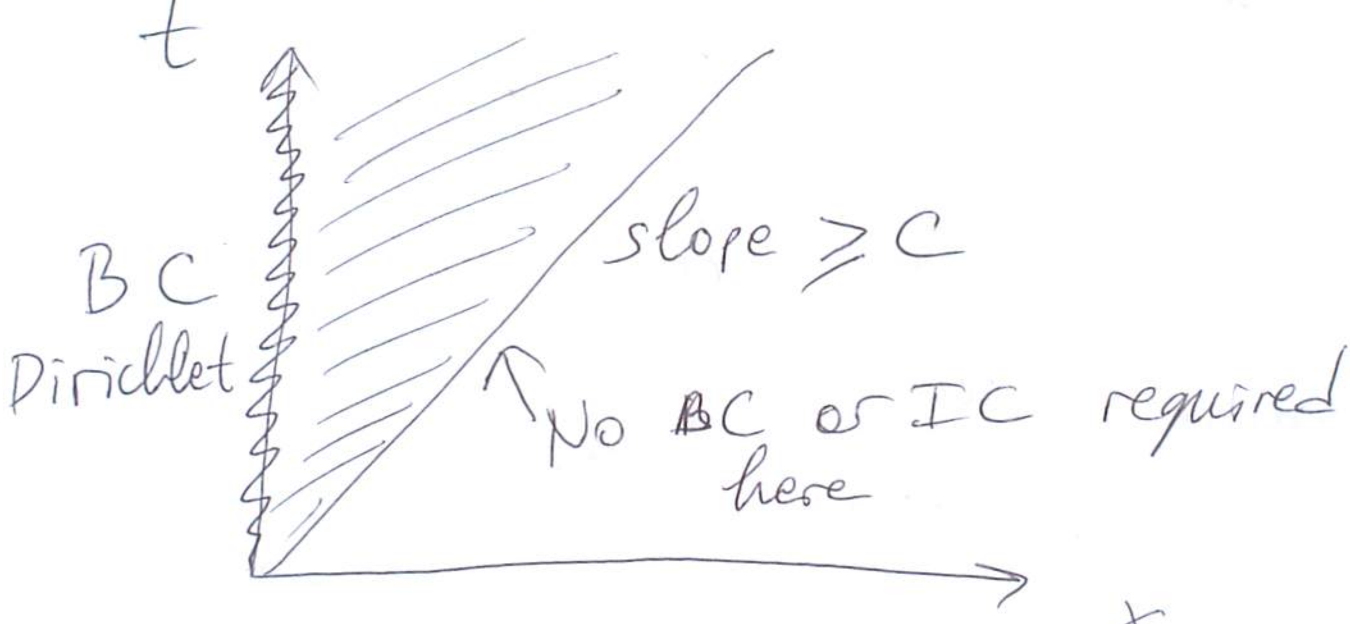
Go back to advection equation

$$u + c u_x = 0$$

$$c > 0$$





Details depend on equation and space-time domain of dependence