PDE Spring 2016

A. Donev

Lecture 17

The Poisson Equations & Laplace

Let's start from the Laplace equation

$$\begin{cases} \nabla^2 u = 0 \text{ in } \Omega \\ u(\partial\Omega) = \varphi(\partial\Omega) \end{cases}$$

↑ Dirichlet BCs

In one-dimension, the Laplace equation is trivial

$$u'' = 0 \Rightarrow$$

$$u = ax + b \text{ (linear)}$$

But in two and three dimensions it is much more interesting

Functions $u(x,y)$ that satisfy

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

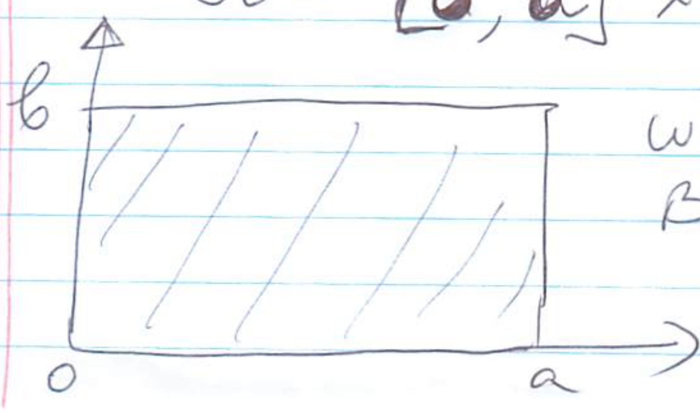
are called <u>harmonic functions</u> and are central to <u>complex analysis</u>

The Laplace equation is often solved by <u>separation of variables</u>

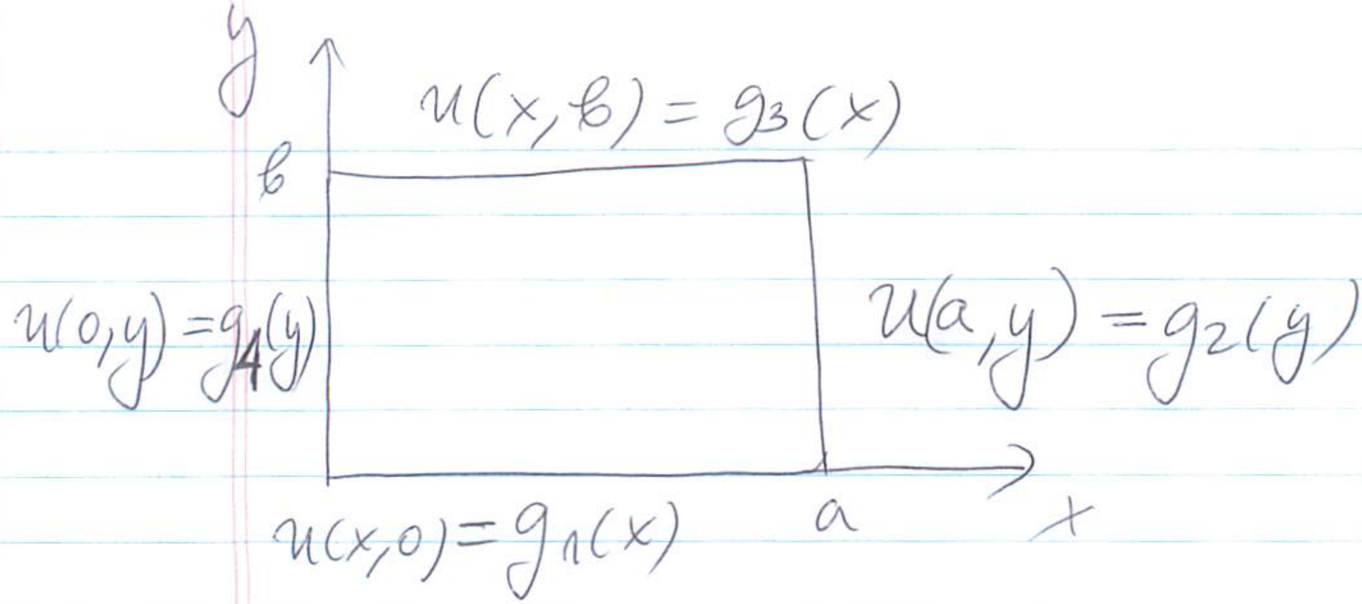
<u>E.g.</u>

Solve $\nabla^2 u = 0$ on rectangle

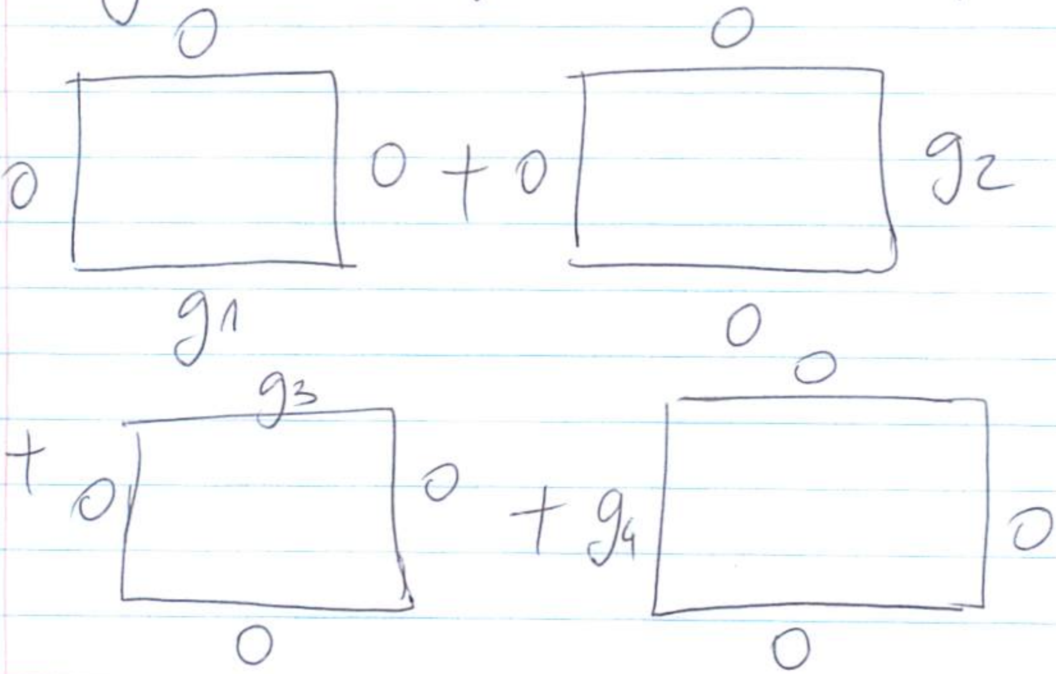
$$\Omega = [0, a] \times [0, b]$$



with inhomogeneous Dirichlet BCs on the sides of the rectangle



We can split this problem into four simpler subproblems by the superposition principle



So it is sufficient to consider non-homogeneous BCs on <u>only one side</u> of the rectangle

$$\begin{cases} \nabla^2 u = 0 \\ u(x,b) = u(a,y) = u(0,y) = 0 \\ u(x,0) = g(x) \end{cases}$$

Look for separable solutions

$$u(x,y) = X(x)\, Y(y)$$

$$\Rightarrow \begin{cases} X(0) = X(b) = 0 \\ Y(b) = 0 \end{cases}$$

but $Y(0)$ is undetermined

set $\boxed{Y(0) = 1}$

$$u_{xx} + u_{yy} = X'' Y + X Y'' = 0$$

$$\Rightarrow -\frac{X''}{X} = \frac{Y''}{Y} = \text{constant} = \lambda$$

We get the one-dimensional eigenvalue problem

$$\begin{cases} X'' = -\lambda X \\ X(0) = X(a) = 0 \end{cases}$$

which we have already solved

$$\begin{cases} X_n = \sin\left(\frac{n\pi x}{a}\right) \\ \lambda_n = \left(\frac{n\pi}{a}\right)^2 \end{cases}$$

$$Y'' = \lambda_n Y(y)$$

positive sign

$$Y = c_1 e^{\sqrt{\lambda_n} y} + c_2 e^{-\sqrt{\lambda_n} y}$$

$$Y(b) = c_1 e^{\sqrt{\lambda_n} b} + c_2 e^{-\sqrt{\lambda_n} b} = 0$$

$$\Rightarrow \quad c_1 = -c_2 e^{-2\sqrt{\lambda_n} b}$$

$$Y = c_2 \left[e^{\sqrt{\lambda_n}(y-2b)} + e^{-\sqrt{\lambda_n} y} \right]$$

Usually written as (6)

$$Y = C_2 e^{-\sqrt{\lambda_n} b}\left[-e^{\sqrt{\lambda_n}(y-b)} + e^{-\sqrt{\lambda_n}(y-b)}\right]$$

Denoting

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

we can write

$$Y = C \sinh\left(\sqrt{\lambda_n}\,(y-b)\right)$$

From

$$Y(0) = 1 = -C \sinh\left(\sqrt{\lambda_n}\, b\right)$$

$$\Rightarrow C = -\frac{1}{\sinh\left(\sqrt{\lambda_n}\, b\right)}$$

Final answer for <u>eigenfunctions</u>

$$\begin{cases} u_n(x,y) = \dfrac{\sinh\left(n\pi(1-y/b)/a\right)}{\sinh\left(n\pi/a\right)} \sin\left(\dfrac{n\pi x}{a}\right) \\ n = 1, 2, \ldots \end{cases}$$

We hope we can expand the solution as a sum of these

$$u = \sum_{n=1}^{\infty} A_n \, u_n(x,y)$$

$$u(x,0) = g(x) \qquad BC$$

$$\Rightarrow \quad g(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)$$

Fourier sine series

$\Rightarrow$

$$u = \sum_{n=1}^{\infty} \left[\frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx\right] \sin\left(\frac{n\pi x}{a}\right) \cdot \left[\frac{\sinh\left(n\pi(1-y/b)/a\right)}{\sinh\left(n\pi/a\right)}\right]$$

is the solution of Laplace's equation

Now let's consider the Poisson equation on a rectangle 8

$$\begin{cases} \nabla^2 u = u_{xx} + u_{yy} = f(x,y) \\ u(\partial\Omega) = 0 \quad \text{on boundary} \end{cases}$$

Idea: To solve $\mathcal{L} u = f$, first find the eigenfunctions of $\mathcal{L}$, and then expand both solution and rhs in that basis

$$u = \sum_n a_n u_n$$

$$f = \sum_n b_n u_n$$

$$\Rightarrow \mathcal{L} u = \sum_n a_n (\mathcal{L} u_n) =$$

$$= \sum \lambda_n a_n u_n = f = \sum b_n u_n$$

$\Rightarrow$ by orthogonality

$$b_n = \lambda_n a_n \Rightarrow \boxed{a_n = \frac{b_n}{\lambda_n}}$$

This assumes that $\lambda = 0$ is not an eigenvalue, which is a sufficient and necessary condition for $\mathcal{L} u = f$ to have a unique solution

$$\Rightarrow u = \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n} u_n$$

From orthogonality, from

$$f = \sum_n b_n u_n$$

$$\Rightarrow b_n = \frac{(u_n, f)}{(u_n, u_n)} \Rightarrow$$

$$= (u_n, f) \text{ is orthonormal basis}$$

$$\boxed{u = \sum_{n=1}^{\infty} \frac{(u_n, f)}{\lambda_n} u_n}$$

The finite-dimensional linear algebra version of this is:

$$Ax = b \Rightarrow x = A^{-1} b$$

$A = U \Lambda U^*$ if A unitarily diagonalizable

$$\Rightarrow A^{-1} = U \Lambda^{-1} U^*$$

since $A A^{-1} = U \Lambda U^* U \Lambda^{-1} U^*$
$= U U^* = I$

$$\Rightarrow x = A^{-1} b = U \Lambda^{-1} (U^* b)$$

We don't usually solve linear systems in $\mathbb{R}^n$ this way since there are faster ways, but for PDEs it is the best way to construct an "inverse Laplacian"

$$u = \nabla^{-2} f \qquad \text{(notation)}$$

The only missing piece is to construct the eigenfunctions of the Laplacian on a rectangle with homogeneous Dirichlet BCs.

$$\nabla^2 u = -\mu u \quad , \quad u(\partial\Omega) = 0$$

μu

Take
$$u = X(x)\,Y(y) \quad \text{separable}$$
(only works because domain is so simple)

$$\Rightarrow \quad -\frac{X''}{X} = \frac{Y''}{Y} - \mu = \lambda$$

$$\Rightarrow \begin{cases} X'' = -\lambda X \\ X(0) = X(a) = 0 \end{cases} \Rightarrow \lambda = \left(\frac{n\pi}{a}\right)^2$$

Now
$$\begin{cases} Y'' = (\mu - \lambda)\,Y \\ Y(0) = Y(b) = 0 \end{cases}$$
shifted eigenvalue

$$\Rightarrow \mu - \lambda = \left(\frac{m\pi}{b}\right)^2$$

$$\Rightarrow \boxed{\mu_{m,n} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} \quad (12)$$

$$u_{m,n} = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

two integer indices

$$f = \sum_{m,n} b_{m,n}\, u_{m,n}$$

$$b_{m,n} = \frac{(f, u_{m,n})}{(u_{m,n}, u_{m,n})} =$$

$$b_{m,n} = \frac{\int_0^b \int_0^a f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy}{\left(\frac{1}{4}ab\right)}$$

and $$u = \sum_{m,n} \frac{b_{m,n}}{\mu_{m,n}}\, u_{m,n}$$

In this case the x and y direction completely separate

Note that the Poisson equation with inhomogeneous BCs is, by superposition, a sum of Laplace with inhomogeneous BCs and Poisson with homog. conditions

$$\begin{cases} \nabla^2 u = \mathcal{L} u = f \\ u(\partial\Omega) = \varphi(\partial\Omega) \end{cases} \Longrightarrow$$

$$u = u_1 + u_2$$

$$\begin{cases} \nabla^2 u_1 = f \\ u(\partial\Omega) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \nabla^2 u_2 = 0 \\ u_2(\partial\Omega) = \varphi \end{cases}$$

We will not prove it here but it is not hard to show that the Poisson & Laplace equations with Dirichlet BCs are well-posed, notably, the solution is unique

(we already proved the maximum principle for Laplace's equation & this can be used in a proof)

A sidenote: Green's Functions

Assume that we could find the Green's function for the Poisson equation with homogeneous BCs

$$\begin{cases} \nabla^2 G = \delta(x - x_0, y - y_0) \leftarrow \text{Point source} \\ (x_0, y_0) \in \Omega \\ G(\partial\Omega) = 0 \quad (\text{homogeneous Dirichlet}) \end{cases}$$

Since

$$f(x,y) = \iint_{(x_0,y_0)\in\Omega} f(x_0,y_0)\,\delta(x-x_0, y-y_0)\,dx_0\,dy_0$$

by superposition $\mathcal{L}u = f \Rightarrow$

$$u(x,y) = \iint_{(x_0,y_0)\in\Omega} f(x_0,y_0)\,G(x,y;x_0,y_0)\,dx_0\,dy_0$$

We usully write this in vector notation

$$G(x,y;x_0,y_0) \equiv G(\underset{\substack{\uparrow \\ \text{target}}}{\vec{x}}, \underset{\substack{\uparrow \\ \text{source}}}{\vec{x}_0})$$

$$u(\vec{x}) = \int_{\Omega} G(\vec{x},\vec{y})\, f(\vec{y})\, d\vec{y}$$

Recall our previous formula

$$u(\vec{x}) = \sum_{n=1}^{\infty} \frac{(f, u_n)}{\lambda_n (u_n, u_n)} u_n(\vec{x})$$

$$= \sum_{n=1}^{\infty} \frac{1}{(\lambda_n \|u_n\|_2^2)} \left[\int_{\Omega} f(\vec{y})\, u_n(\vec{y})\, d\vec{y} \right] u_n(\vec{x})$$

$$= \int_{\Omega} \underbrace{\left(\sum_{n=1}^{\infty} \frac{u_n(\vec{y})\, u_n(\vec{x})}{\lambda_n \|u_n\|_2^2} \right)} f(\vec{y})\, d\vec{y}$$

<u>Defines</u> $G(\vec{x},\vec{y})$