# PDE Spring 2016

A. Donev

## Lecture 16

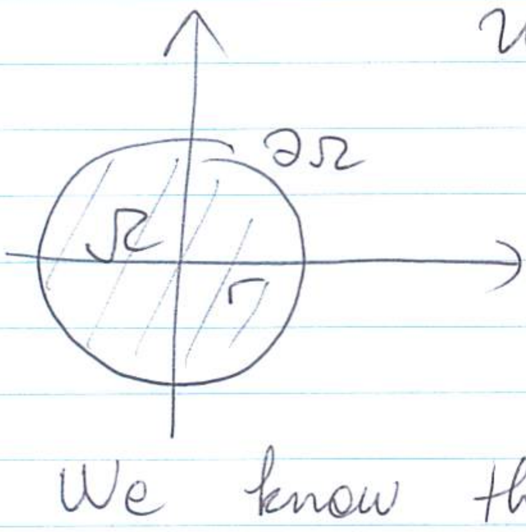
## Sturm-Liouville Problems

As motivation let us consider the heat equation on a circle with a <u>radially-symmetric IC & BC</u>

E.g.

$$u_t = u_{xx} + u_{yy} = \nabla^2 u$$

for $x^2+y^2<r^2$




$$u(\partial\Omega) = 0$$

$$u(\Omega, t=0) = \varphi(\Omega)$$

IC

We know that the solution is spherically-symmetric so we should use <u>polar coordinates</u>

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad (2)$$

$$-\nabla^2 u = -\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

(from Calculus 3)

For us $u \equiv u(r)$ only, so the eigenvalue problem we need to solve is

$\mathcal{L} u = \lambda u$ where

$$(\mathcal{L} u)(r) = -\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

$$= -\frac{1}{r}(r u')' = \lambda u$$

So we get

$$\begin{cases} \frac{1}{r}(r u')' + \lambda u = 0 \\ u(0) = u(r) = 0 \end{cases}$$

which is a one-dimensional problem (same in 3D but different coefficients)

These types of problems appear often in practice and we consider them in this class. ③

We consider the <u>two-point BVP</u> (i.e. ODE!)

$$\boxed{\mathcal{L}u = -(p(x)u'(x))' + q(x)u}$$

$$= -pu'' - p'u' + qu = f(x)$$

on $a \leq x \leq b$, with either Dirichlet, Neumann, periodic, or mixed BCs

There are two cases to be considered:

① <u>Non-singular</u>:

$p(x) > 0$, $q(x) > 0$ $\forall a < x < b$

cont. differentiable   continuous

② <u>Singular</u> (harder!)

$p(a) = 0$ or $p(b) = 0$

Observe that a general BVP in one dimension that is linear and second-order takes the form

$\times p(x)$ $$u'' = a(x)u' + b(x)u - g(x)$$

With a trick we can write this in the SL form:

$$p(x)u'' - p(x)a(x)u' - p(x)b(x)u = p(x)g(x)$$

From $(pu')' = pu'' + p'u'$ we get $pu'' = (pu')' - p'u'$

$\Rightarrow$

$$(pu')' - (pa + p')u' - pbu = pg$$

versus the SL equation

$$(pu')' \qquad\qquad + qu = f(x)$$

So we want $pa + p' = 0$

(5)

$$p' = -a(x)p \Rightarrow$$

$$\begin{cases} p(x) = \exp\left[-\int a(x)\,dx\right] \\ q(x) = p(x)\,b(x) \\ f(x) = -p(x)\,g(x) \end{cases}$$

Observe that as long as $a(x)$ is <u>integrable</u>, $p(x)$ exists and is positive, so the Sturm-Liouville problem is quite general!

The key observation that will allow us to solve SL problems is to find the <u>SL eigenfunctions</u> & eigenvalues

$$\mathcal{L}u = \lambda u \quad + \text{BCs}$$

and then expand the rhs $f(x)$ or initial condition for IVPs into an <u>orthogonal series</u>

# Eigenfunctions

The key observation is that:

$\left\{\begin{array}{l}\text{The SL operator is an} \\ \text{self-adjoint operator}\end{array}\right.$

$$(u, \mathcal{L}v) = \int_a^b u\left[-(p\bar{v}')' + q\bar{v}\right]dx$$

$$= -\int_a^b \underbrace{u\,(p\bar{v}')'}_{\text{integrate by parts}}dx + \int_a^b q u \bar{v}\, dx$$

$$= -\left[p u v\right]_a^b$$

$$+ \int_a^b \left((pu')\bar{v}' + q u v\right) dx$$

integrate by parts again

$$= \left[p(u'\bar{v} - u\bar{v}')\right]_a^b + (\mathcal{L}u, v)$$

Therefore (7)

$$(u, \mathcal{L}v) - (\mathcal{L}u, v) = \underbrace{\left[p(u'\bar{v} - u\bar{v}')\right]_a^b}_{\text{this vanishes for many BCs}}$$

So for a number of common BCs we have

$$(u, \mathcal{L}v) = (\mathcal{L}u, v) \Rightarrow$$

$$\mathcal{L}^* = \mathcal{L} \rightarrow \text{self-adjoint}$$

From the fact $\mathcal{L}$ is a symmetric (self-adjoint) operator we already know a lot of consequencies.

E.g. it's eigenvalues are real, eigenvectors are orthogonal (even complete), etc.

Turns out we know their sign also.

Theorem: 8

If $p(x) > 0$, $q(x) \geq 0$ for $x \in (a, b)$, then the eigenvalues are real and positive

→ this means the operator is symmetric positive definite

Since $\lambda = 0$ is not an eigenvalue, we know

$$\mathcal{L} u = 0$$

has only $u = 0$ as the solution. This means

$$\mathcal{L} u = f(x)$$

has a unique solution — why?

i.e. the SL two-point BVP is well-posed.

To prove $\lambda$ are real we only need $\mathcal{L}^* = \mathcal{L}$ and we proved this already.

To prove they are positive take

$$(u, \mathcal{L} u) = \lambda (u, u)$$

$$= \int_a^b \left( - \bar{u} (p u')' + q \bar{u} u \right) dx$$

integrate by parts

$$= \int_a^b \left( p |u'|^2 + q |u|^2 \right) dx \geq 0$$

$+$ vanishing boundary terms

$$\Rightarrow \lambda = \frac{\int_a^b (p|u'|^2 + q|u|^2) dx}{\int |u|^2 dx}$$

$$\Rightarrow \boxed{\lambda > 0}$$

We in fact know a few more things (not proven here): 

1. The eigenvalues are <u>simple</u> (not repeated) and there are countably many of them with

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots\ldots$$

and $\lambda_k \to +\infty$ for $k \to \infty$

2. All eigenvectors are orthogonal (we have proven this)

3. The eigenfunctions form a complete $L_2$ basis, i.e.,

$\forall f(x) \in L_2$

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x) \qquad c_n = \frac{(u_n, f)}{(u_n, u_n)}$$

convergence in norm