PDE Spring 2016

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Lecture 14

Fourier / Orthogonal Series

In order for the method of separation of variables to work, we must be able to expand the initial condition in a Fourier series

$$0 \le x \le \ell$$

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$

(for heat equation with homogeneous Dirichlet conditions)

This opens a number of questions from (harmonic) analysis: ②

① For what $\varphi(x)$ is this possible?

② If possible, how do we find the coefficients A_n

③ What does the equal sign really mean, since the sum is infinite

④ Is this approach restricted to heat equation or general?

It turns out we will get most understanding and depth from #4

Recall that the way we got the sin functions was not arbitrary - we solved an eigenvalue BVP. So let us first generalize linear algebra to functions.

Function Vector Spaces

Function spaces are *infinite dimensional* vector spaces where the role of vectors is played by *functions*

One can also think of them as *classes* of functions closed under scalar multiplication and addition.

Examples: of vector spaces of functions

1. Space of all *polynomials* of degree n, P_n
2. Space of all polynomials P
3. Space of all continuous functions C^0
4. Continuously differentiable functions C^1
5. Twice contin. diff. C^2
6. Infinitely differentiable C^∞ (smooth)
7. Analytic functions

We consider here functions defined on interval $[a, b]$

<u>Inner product</u> for functions

$$(f, g) = \int_a^b f(x)\,\bar{g}(x)\,dx$$

complex conjugate

<u>Norms</u> for functions

L_∞: $\|f\|_\infty = \max_{a \le x \le b} |f(x)|$

or max norm

L_1: $\|f\|_1 = \int_a^b |f(x)|\,dx$

Manhattan norm

L_2: $\|f\|_2 = \left[\int_a^b |f(x)|^2\,dx\right]^{1/2}$

Eucledian norm

Note that these norms are all distinct (unlike for vectors where they are an "arbitrary" choice)

A basis set for a linear function space is a collection of functions such that (5)

$\forall f \in V$

$$f(x) = \sum_{n=1} c_n f_n(x)$$

Basis is $\{ f_1(x), f_2(x), \ldots \}$

<u>Example</u>

Space of polynomials of degree n is spanned by the <u>monomial basis</u>

$$\{ x^0, x^1, x^2, x^3, \ldots, x^n \}$$

Two functions are orthogonal if $(f_1, f_2) = 0$

An <u>orthonormal basis</u> satisfies

$$(f_i, f_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Essentially all of the linear algebra review we did for matrices (finite-dimensional vector spaces) generalizes to linear operators (infinite-dimensional operators).

Key differences

1. The number of elements (vectors) in the basis sets will be countably infinite rather than finite - so there is no equivalent of $\mathbb{R}^n$ / $\mathbb{C}^n$

2. ~~[illegible]~~ There is no generic vector space / basis (as is $\mathbb{R}^n$ for n-dimensional spaces), rather, there are different functional spaces and all norms are different.

3. Therefore, to really specify a PDE one must specify the function space / inner product / norm one is interested in.

Theorem

The Laplacian with Dirichlet/Neumann BCs ~~is~~ is a Hermitian or self-adjoint operator

More precisely, let us consider the space of all C^2 functions (twice continuosly differentiable) that satisfy one of the conditions

$$\begin{cases} \underline{\overline{X}}(0) = 0 & \overline{X}(\ell) = 0 \\ \text{or } \underline{\overline{X}}_x(0) = 0 & \underline{\overline{X}}_x(\ell) = 0 \end{cases}$$

Now consider the Laplacian linear operator acting on functions in this vector space.

Recall definition of self-adjoint:

$$(X_1, \mathcal{L} X_2) = (\mathcal{L} X_1, X_2)$$

$$\mathcal{L} \equiv \partial_{xx}$$

Let us work with real numbers for now (not crucial but we will see it is sufficent)

$$(X_1, \mathcal{L} X_2) = \int_0^{\ell} X_1 X_2'' \, dx =$$

integrate by parts

$$= -\int_0^{\ell} X_1' X_2' \, dx + \left[X_1 X_2' \right]_0^{\ell}$$

integrate by parts again

$$= \int_0^{\ell} X_1'' X_2 \, dx - \left[X_1' X_2 \right]_0^{\ell} + \left[X_1 X_2' \right]_0^{\ell}$$

Therefore

$$\int_0^{\ell} \left(X_1 X_2'' - X_1'' X_2 \right) dx = \left[X_1 X_2' - X_1' X_2 \right]_0^{\ell}$$

which is called Green's second identity

In operator notation

$$(X_1, \mathcal{L} X_2) - (\mathcal{L} X_1, X_2) = \left[X_1 X_2' - X_1' X_2 \right]_0^{\ell}$$

Now we see that for any combination of Dirichlet or Neumman conditions the boundary term vanishes, and so

$$(X_1, \mathcal{L} X_2) = (\mathcal{L} X_1, X_2)$$

i.e. $\mathcal{L}$ is self-adjoint

Theorem:

Every self-adjoint linear operator possesses a set of orthogonal eigenvectors with real eigenvalues. eigenfunctions

In fact, for the Laplacian all eigenvalues are non-positive

To see this, take an eigenfunction $X(x)$ and see

$$(X, \mathcal{L} X) = \int X \, X'' \, dx$$

$$= -\int (X')^2 \, dx + \left[X X' \right]_0^{\ell}$$

$$= -\int (X')^2 \, dx \leq 0$$

$$= (X, \lambda X) = \lambda (X, X)$$

$$\Rightarrow \quad \lambda = \frac{-\int (X)' \, dx}{\int X^2 \, dx} \leq 0$$

Observe that all of these proofs rely only on the axioms and thus generalize from ordinary linear algebra...

We now know that the $\sin\left(\frac{n\pi x}{\ell}\right)$ functions are orthogonal!

This can be proven using

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

but we don't have to prove it again!

If we take the Laplacian with Neumann BCs, we get:

$$\begin{cases} X'' = -\lambda X \\ X'(0) = X'(\ell) = 0 \end{cases}$$

proceding as for Dirichlet (do it on your own!), we get

$$\begin{cases} \lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \quad \text{for } n = 0, 1, 2 \\ X_n = \cos\frac{n\pi x}{\ell} \end{cases}$$

Note $\emptyset$ is here!

and of course these cos functions must also be orthogonal to each other.

The separation of variables method for

$$\begin{cases} u_t = k u_{xx} & u(x,0) = \varphi(x) \\ u_x(0) = u_x(\ell) = 0 \end{cases}$$

gives

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-(n\pi/\ell)^2 kt} \cos\left(\frac{n\pi x}{\ell}\right)$$

with A_n determined from

$$\varphi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right)$$

which is called the

<u>Fourier cosine series.</u>

(The factor $\frac{1}{2}$ will be explained later)

Assume that we have an orthogonal basis set of functions, and that

$$\psi(x) = \sum_{n=0,1 \text{ or } -\infty}^{\infty} A_n f_n(x)$$

How do we find A_n?

Follow the same procedure as for representing in a unary basis

$$x = \mathcal{U} y \Rightarrow y = \mathcal{U}^{-1} x = \mathcal{U}^{*} x$$

$$\psi(x) = \sum_n A_n f_n$$ Multiply by f_m

$$(f_m, \psi) = \sum_n A_n (f_m, f_n)$$

$$= \sum_n A_n \delta_{mn} (f_n, f_n)$$

$$= A_m (f_m, f_m) \Rightarrow$$

$$\boxed{A_n = \frac{(f_m, \psi)}{(f_m, f_m)}}$$ general!

Let's apply this to the sine series

$$f_n = \sin\left(\frac{n\pi x}{\ell}\right)$$

$$(f_n, f_n) = \int_0^\ell \sin^2\left(\frac{n\pi x}{\ell}\right) dx = \frac{\ell}{2}$$

$\Rightarrow$

$$\boxed{\begin{aligned} \varphi(x) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right) \Rightarrow \\ A_n &= \frac{2}{\ell} \int_0^\ell \varphi(x) \sin\left(\frac{n\pi x}{\ell}\right) dx \end{aligned}}$$

Fourier sine series

Similarly, for cosine series

$$\boxed{\begin{aligned} \varphi(x) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right) \Rightarrow \\ A_n &= \frac{2}{\ell} \int_0^\ell \varphi(x) \cos\left(\frac{n\pi x}{\ell}\right) dx \end{aligned}}$$

and we put the factor $1/2$ in front of A_0 so this covers $n=0$

Periodic boundary conditions

As a last example consider the heat equation with <u>periodic BCs</u>:

$$\begin{cases} u(0) = u(l) \\ u_x(0) = u_x(l) \end{cases}$$

$\leftarrow$ useful in physics of waves

The eigenvalue problem is

$$\begin{cases} X'' = -\lambda X \\ X(0) = X(l) \\ X'(0) = X'(l) \end{cases}$$

Recall $X(x) = e^{-\lambda x}$ and we want

$$e^{-0} = 1 = e^{-\lambda l}$$

$$\lambda = a + ib$$

$$e^{-(a+ib)l} = e^{-al} e^{-ibl} = 1$$

Note $|e^{-al}| \leq 1$ and

$$|e^{-ibl}| = |\cos(bl) + i\sin(bl)| = 1$$

So $|e^{-al}e^{-ibl}| = |e^{-al}||e^{-ibl}| \leq 1$

with $= 1$ only if $a = 0$

So λ must be <u>purely imaginary</u>

$$e^{ibl} = \underbrace{\cos(bl)}_{=1} + i\underbrace{\sin(bl)}_{=0} = 1$$

$$\Rightarrow b = \frac{2n\pi}{l}, \quad n = \text{integer}$$

To avoid the factor of 2 usually the problem is specified on the interval $(-l, l)$

$$u(-l) = u(l)$$

$$\Rightarrow \boxed{X(x) = e^{\frac{in\pi x}{l}}, \quad n \in \mathbb{Z}}$$

which is often simpler to work with. These are <u>orthogonal functions</u>

$$(\underline{X}_n, \underline{X}_n) = \int_{-\ell}^{\ell} e^{in\pi x/\ell} e^{-in\pi x/\ell} dx$$

$$= \int_{-\ell}^{\ell} 1 \, dx = 2\ell$$

$$\Rightarrow \boxed{\begin{aligned} \varphi(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell} \Rightarrow \\ c_n &= \frac{1}{2\ell} \int_{-\ell}^{\ell} \varphi(x) e^{-in\pi x/\ell} dx \end{aligned}}$$

Complex / Periodic Fourier series

Note that one can rewrite this to use real numbers only but analytical computations are easier using the complex notation

In fact,

$$\begin{cases} \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2} \\ \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} \end{cases}$$

simplifies a lot of algebra often