

PDE Spring 2016 (1)
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Lecture 13

Fourier Series part 1
(Review)

Before we begin Fourier series I would like to review some abstract but key concepts from linear algebra

A vector space V is a set of elements $\vec{x} \in V$ such that

① Scalar multiplication

$$\alpha \vec{x} \in V$$

② Vector addition

$$\vec{z} = \alpha \vec{x} + \beta \vec{y} \in V$$

A subspace $V' \subseteq V$ is a subset that is closed under scalar multiplication and addition, i.e., under linear combinations.

$$\text{E.g. } \mathbb{R}^3 \equiv V \quad (2)$$
$$V' = \{ \vec{x} \in \mathbb{R}^3 \mid x_3 = 0 \} \sim \mathbb{R}^2$$

A set of vectors is linearly independent or forms a basis if there is no nontrivial linear combination equal to zero

$$\sum_{i=1}^n \alpha_i \vec{a}_i = 0 \Rightarrow \alpha_i = 0$$

The dimension of a vector (sub) space is the number of elements in a basis

$$\dim V = n$$

e.g. $\dim \mathbb{R}^n = n$

Every $\vec{z} \in V$ can be uniquely represented in a given basis

$$\vec{z} = \sum_{i=1}^n \alpha_i \vec{a}_i$$

③

That is, a vector can be represented as a list of n coefficients in a given basis:

$$\vec{z} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

For \mathbb{R}^m the usual assumed basis is

$$\alpha_1 = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \alpha_2 = e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ etc.}$$

If we collect a set of m vectors we get a matrix

$$A = \begin{bmatrix} | & | & \dots & | \\ a_1 & a_2 & \dots & a_m \\ | & | & \dots & | \end{bmatrix}$$

↑
columns of A

A matrix-vector product is

$$b = Ax = \sum_{i=1}^m x_i a_i$$

Observe that Ax is ④
an element of some other
vector space, called image or

Therefore range or column space
of matrix A

$$y = Ax$$

defines a linear mapping or
transformation (operator) from
one (finite-dimensional) vector
space to another.

The actual coefficients depend on
the choice of basis (both for
the domain and the image of
the mapping) but the key
concept is the mapping itself.

Linearity means that

$$\left. \begin{array}{l} A(\alpha x) = \alpha Ax \\ A(x+y) = Ax + Ay \end{array} \right\}$$

as we already explained

Every linear transformation
can be represented by a matrix

$$A: \mathcal{V} \rightarrow \mathcal{V}' \quad \text{⑤}$$

domain
column space / range / image

The dimension of the column space of A is the rank

$$r = \text{rank } A = \dim \mathcal{V}'$$

$$\boxed{r \leq \min(m, n)}$$

If $r = \min(m, n)$ the matrix is full rank

The null space of a matrix is the set of solutions of

$$Ax = 0$$

$$\text{null } A = \{x \mid Ax = 0\}$$

This is also a subspace $\mathcal{V}' \subseteq \mathcal{V}$
(prove it)

nullity $A = \dim(\text{null } A)$
is the number of linearly independent solutions of $Ax = 0$

Fundamental theorem of LA:

(6)

rank + nullity = n
for $A \equiv [m \times n]$ matrix
i.e. $A \in \mathbb{R}^{m,n}$, or $\mathbb{C}^{m \times n}$

Most of the time we are interested in one vector space and mappings from it to itself, i.e., $n \times n$ square matrix

A square matrix forms a basis or is invertible iff

$$\exists B \equiv [n \times n] \text{ s.t. } AB = BA = I$$

where I is the identity matrix

$$I_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} = \delta_{ij}$$

Kronecker δ

$$B \equiv A^{-1}$$

The following are equivalent: (7)

- (1) A is invertible
- (2) A is full-rank, $\text{rank } A = n$
- (3) Columns of A are a basis
- (4) $\det A \neq 0$
- (5) Zero is not an eigenvalue of A , i.e., there is no non-zero solution of $Ax = 0$

Expressing one vector in a given basis means solving the linear system

$$Ax = y \leftarrow \text{vector}$$

\uparrow
coefficients

If A is invertible a solution always exists,

$$x = A^{-1}y$$

Orthogonality

(8)

A combination of a vector space with an inner or dot product gives an inner-product space.

Inner product is a map from two elements of V to a scalar (element of a field)

$$(x, y) \equiv \langle x, y \rangle \equiv x \cdot y$$

$$(x, y) : V \times V \rightarrow \mathbb{C}$$

and satisfies: the three axioms:

① $(x, y) = \overline{(y, x)}$ conjugacy
or symmetry

② Linearity in first argument

$$(\alpha x, y) = \alpha (x, y)$$

$$(x + y, z) = (x, z) + (y, z)$$

③ Positive definiteness

⑨

$(x, x) \geq 0$ and real
and $(x, x) = 0$ iff $x = 0$

Standard inner product

$$(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y} = \sum_{i=1}^n x_i y_i$$

Two vectors are orthogonal if

$$(\vec{x}, \vec{y}) = 0$$

real $\|\vec{x}\|^2 = (\vec{x}, \vec{x}) = \vec{x} \cdot \vec{x} \geq 0$

L_2 norm or length of vector
better denoted with $\|\vec{x}\|_2$

$$\|\vec{x}\|_2 = (\vec{x}, \vec{x})^{1/2}$$

A vector is normalized if

$$\|\vec{x}\| = 1$$

An adjoint of a linear operator is defined by (10)

$$B = A^* \text{ if}$$

$$\boxed{(Ax, y) = (x, A^*y)}$$

For finite dimensional matrices, this is just the conjugate transpose for complex matrices

$$A^* = \overline{A}^T \leftarrow \text{transpose}$$

or for real matrices just the transpose.

But the above definition is the more general one.

Two vector spaces are orthogonal if

$$V \perp V' \text{ if } \forall x \in V, \forall y \in V' \quad (x, y) = 0$$

E.g. $\{z \in \mathbb{R}^3 \mid z_1 = 0, z_2 = 0\}$
and $\{z \in \mathbb{R}^3 \mid z_3 = 0\}$

A matrix is unitary if 12
its columns are normalized
and orthogonal to each other.

This implies

$$\boxed{U^{-1} = U^*}$$

For real matrices usually the
word orthogonal matrix is used.

Recall the problem of expressing
a vector y in an orthonormal
basis given by the columns
of U :

$$\begin{aligned} U \vec{x} &= \vec{y} \Rightarrow \\ \vec{x} &= U^{-1} \vec{y} = U^* \vec{y} \end{aligned}$$

$$\Rightarrow \boxed{x_i = (\vec{y}, \vec{u}_i)}$$

orthogonal projection onto each
column of U .

How to see this quickly

$$\vec{y} = \sum_i x_i \vec{u}_i$$

Dot both sides by \vec{u}_j

$$\vec{y} \cdot \vec{u}_j = \sum_i x_i (\vec{u}_i \cdot \vec{u}_j)$$

$$= \sum_i x_i \delta_{ij} = x_j$$

$$\Rightarrow x_j = \vec{y} \cdot \vec{u}_j$$

as we claimed.



More general vector norm:

$$L_p \text{ norm: } \|x\|_p = \left(\sum_{i=1}^N |x_i|^p \right)^{1/p}, p \geq 1$$

① $p=1$, L_1 norm

$$\|x\|_1 = \sum_i |x_i|$$

② $p=2$, L_2 norm

$$\|x\|_2 = (x, x)^{1/2} = \sqrt{\sum_{i=1}^N x_i^2}$$

③ $p=\infty$, $\|x\|_\infty = \max_i |x_i|$

A norm (vector norm) is itself a well-defined mathematical concept. (13)

It is a mapping from $V \rightarrow \mathbb{R}$ vectors to real numbers

axioms

$$\left\{ \begin{array}{l} \textcircled{1} \quad \|\alpha \vec{v}\| = |\alpha| \|\vec{v}\| \\ \textcircled{2} \quad \|\vec{v} + \vec{u}\| \leq \|\vec{v}\| + \|\vec{u}\| \\ \quad \quad \quad \text{(triangle inequality)} \\ \textcircled{3} \quad \exists \vec{v} \quad \|\vec{v}\| = 0 \Leftrightarrow \vec{v} = \vec{0} \end{array} \right.$$

These imply $\left\{ \begin{array}{l} \|\vec{v}\| \geq 0 \\ \text{non-negativity} \end{array} \right.$

So we can think of the norm as a generalization of length of a vector.

Eigenvalues

For every square matrix \exists at least one λ s.t.

(14)

$$Ax = \lambda x, \quad x \neq 0$$

\uparrow \nwarrow
eigenvector eigenvalue

A matrix is non-defective or diagonalizable if it has n linearly independent eigenvectors.

Form $X = [\vec{x}_1 \mid \vec{x}_2 \mid \dots \mid \vec{x}_n]$

and X is invertible.

Then

$$AX = X\Lambda$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

is a diagonal matrix of eigenvalues.

This means

(15)

$$AX = X\Lambda \quad \left| \begin{array}{l} \text{Multiply} \\ \text{on right} \\ \text{by } X^{-1} \end{array} \right.$$

$$AI = A = X\Lambda X^{-1}$$

$$\boxed{A = X\Lambda X^{-1}}$$

is the eigenvalue decomposition

If we want to multiply a vector by the matrix:

$$Ay = X\Lambda(X^{-1}y) = X\Lambda\tilde{y}$$

Observe that

$$X^{-1}y = \tilde{y} \Rightarrow y = X\tilde{y}$$

that is, \tilde{y} is just y expressed in the basis of eigenvectors.

$$z_i = (\Lambda\tilde{y})_i = \lambda_i \tilde{y}_i$$

so that in this basis the matrix is diagonal (16)

Now $Ay = Xz$
is simply z expressed back
in the original basis.

The eigenvectors give us a
special basis in which the
matrix (linear transformation)
is diagonal
(this is the key concept)

A matrix is normal if

$$AA^* = A^*A$$

Theorem: A is normal iff
one can find an orthonormal
set of eigenvectors that forms
a basis:

$$A = U \Lambda U^{-1} = U \Lambda U^*$$

unitary matrix

$$A = U \Lambda U^*$$

(17)

is the best representation of a normal matrix for many purposes.

A matrix is Hermitian if

$$A^* = A, \text{ or } \underline{\text{self-adjoint}}$$

called symmetric in the real case,

$$A^T = A$$

Since $A^* A = A^2 = A A^*$ the matrix is also normal and so

Every Hermitian matrix is unitarily diagonalizable and all its eigenvalues are real

We will see how to generalize these properties to functions (as vectors) and linear operators (as matrices) next time

Theorem: Eigenvectors corresponding to distinct eigenvalues of a Hermitian / symmetric matrix are orthogonal.

Proof:
$$\begin{cases} Ax_1 = \lambda_1 x_1 \\ Ax_2 = \lambda_2 x_2 \end{cases}$$

$$\begin{aligned} (x_1, Ax_2) &= (x_1, \lambda_2 x_2) = \lambda_2 (x_1, x_2) \\ &= (A^* x_1, x_2) = (Ax_1, x_2) = (\lambda_1 x_1, x_2) \\ &= \lambda_1 (x_1, x_2) \end{aligned}$$

$$\Rightarrow (\lambda_1 - \lambda_2) (x_1, x_2) = 0$$

Since $\lambda_1 \neq \lambda_2 \Rightarrow (x_1, x_2) = 0$
as needed to show

Note: It can also be shown that eigenvectors corresponding to the same eigenvalue can be made orthogonal (Gram-Schmidt procedure)

Similar proofs can be done to show that eigenvalues are real:

Recall adjoint property of dot product:

$$\overline{(x, y)} = (y, x)$$

$$\Rightarrow \overline{(Ax, x)} = (Ax, x)$$

$$\overline{\lambda(x, x)} = \lambda(x, x)$$

Since $(x, x) > 0$ and real (recall $(x, x) = \|x\|_2^2$),

we get

$$(\overline{\lambda} - \lambda)(x, x) = 0 \Rightarrow$$

$$\overline{\lambda} = \lambda, \text{ i.e., } \lambda \text{ is real}$$

Conclusion: Every Hermitian matrix is unitarily diagonalizable with real eigenvalues and real eigenvectors (complex numbers not necessary)