PDE Spring 2016

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Lecture 12

Separation of Variables

In this first lecture of the second half of the semester we will motivate the rest by looking at the heat equation on a *bounded domain*:

$$
\begin{cases}
u_t = k u_{xx} \\
0 < x < 1 \quad , \quad t > 0 \\
u(x,0) = g(x) \qquad \text{IC} \\
u(0,t) = u(1,t) = 0 \quad \left\} \begin{array}{l} \text{homogeneous} \\ \text{Dirichlet} \\ \text{BC} \end{array} \right.
\end{cases}
$$

Where we do *not* require that $g(1) = g(0) = \underline{1}$, allowing the solution to start discontinu

One way to solve this would be to find the Green's function by setting

$$g(x) = \delta(x - x_0)$$

But this is almost as hard as the original problem because of the <u>nontrivial BCs</u>.

Let us try the <u>ansatz</u> (educated guess)

$$u(x,t) = X(x)\, T(t)$$

function <u>only</u> of x

Function only of t

This means the two independent <u>variables are separated</u>.

It will take us several lectures to understand why this was a good guess...

But for now let's try it

BCs give:

$$\begin{cases} u(0,t) = X(0)\,T(t) = 0 \\ u(1,t) = X(1)\,T(t) = 1 \end{cases}$$

This means (since $T \neq 0$)

$$\boxed{X(0) = X(1) = 0}$$

i.e. the <u>spatial part satisfies the BCs</u>

Now, <u>plug</u> into PDE

$$u_t = X(x)\,T_t(t)$$

$$k\,u_{xx} = k\,X_{xx}(x)\,T_t(t)$$

$$\Rightarrow \quad \underbrace{\frac{1}{kT(t)}T'(t)}_{\text{Function of } t} = \underbrace{\frac{1}{X(x)}X''(x)}_{\text{Function of } x}$$

Since two functions of two different variables are equal, they <u>must</u> both be constant:

$$\boxed{\frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = -\lambda}$$

Sign for later $\lambda > 0$

This is now a system of two ODEs!

The easier one to solve is

$$T' = -\lambda k T \Rightarrow$$

$$\boxed{T(t) = T(0)\exp(-\lambda k t)}$$

and this exponential decay is typical of the heat equation

The harder equation is

$$\text{BVP}\quad \begin{cases} X'' = -\lambda X \\ X(0) = X(1) = 0 \end{cases}$$

This is called a <u>two-point boundary value problem</u>

The trivial solution is
$$\bar{X}(x) = 0$$
but we want nontrivial ones.

The general solution has the form
$$\bar{X} = a e^{\alpha x}$$
where $\alpha^2 = -\lambda$ from the ODE
Since the sign of α is arbitrary
$$\bar{X} = a e^{\alpha x} + b e^{-\alpha x}$$
is the general solution (we need two constants for second-order)

$$\bar{X}(0) = a + b = 0$$
$$\bar{X}(1) = a e^{\alpha} + b e^{-\alpha} = 0$$
$$= a(e^{\alpha} - e^{-\alpha}) = 0$$
If $a = 0 \Rightarrow b = 0$ so trivial
If $a \neq 0$ then
$$e^{\alpha} - \frac{1}{e^{\alpha}} = 0 \Rightarrow (e^{\alpha})^2 = \underline{1}$$

Now let's assume that 6

$\lambda < 0$

$\alpha^2 = -\lambda > 0$ and $\alpha > 0$ is <u>real</u>

$e^{\alpha} = +1 \Rightarrow \alpha = 0$

which is a contradiction

If $\lambda = 0$ then equation is

$$X'' = 0 \Rightarrow X(x) = ax + b$$

and $X(0) = X(1)$ forces $a = b = 0$
so we get the trivial solution

But note this is a <u>particular solution</u> if nonhomog. BCs

We conclude

$$\boxed{\lambda > 0}$$

$\alpha^2 = -\lambda < 0$ so $\alpha = i\beta$ is <u>complex</u>

$$e^{i\beta x} = \cos(\beta x) + i \sin(\beta x)$$

and so it is better to switch now to sines and cosines

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$\bar{X}(x) = A\cos\beta x + B\sin\beta x$

$\bar{X}(0) = 0 \Rightarrow A = 0$

$\bar{X}(1) = B\sin(\beta x) = 0 \Rightarrow$

$\sin(\beta x) = 0$

since $B = 0$ is trivial solution.

$\Rightarrow \beta =$ integer, positive

$\beta = n\pi \in \mathbb{Z}^+$

$\alpha^2 = -\beta^2 = -\lambda \Rightarrow$

$\boxed{\lambda = n^2\pi^2}$ where $n \in \mathbb{Z}^+$

This is the fundamental result of the separation of variables:

The set of solutions of

$\mathcal{L}u = 0, \mathcal{B}u = 0$

is countable and can be enumerated by positive integers.

Sidenote: Compare this to the case of finite-dimensional 8

$$Ax = 0$$

Which has a finite number of independent solutions, called the nullity of the matrix

$$X(x) = B \sin(n\pi x)$$

$$\Rightarrow \boxed{u_n(x,t) = B_n e^{-kn^2\pi^2 t} \sin(n\pi x)}$$

We have therefore identified a family of countably infinetely many solutions.

Claim: Any solution can be be expressed as a sum of these

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-kn^2\pi^2 t} \sin(n\pi x)$$

How to determine B_n?

The one thing we have not used is the initial condition: (9)

$$u(x,0) = \boxed{\sum_{n=1}^{\infty} B_n \sin(\pi n x) = g(x)}$$

The question now becomes:

For what functions $g(x)$ does a convergent series of the above form exist? When it exists, is it unique, and if so, how to determine the B_n coefficients?

To answer this question we will need to go into <u>Fourier analysis</u>

It turns out almost every function can be expanded in this way and

$$B_n = 2 \int_{x=0}^{1} g(x) \sin(n\pi x)\, dx$$

as we will obtain later.

Some comments:
the problem

$$\underline{X}'' = -\lambda \underline{X} \quad , \underline{X}(0) = \underline{X}(1) = 0$$

is an <u>eigenvalue problem</u> very similar to the matrix $Ax = \lambda x$

The λ's that are possible are the <u>eigenvalues</u> and

$$\underline{X}(x) = \sin(n\pi x)$$

are the <u>eigenfunctions</u>.

Many of the concepts from linear algebra will generalize, so we will review some of those in the upcoming lectures.

Also observe that the <u>high-frequency</u> modes (large n) <u>decay very rapidly</u> $(\exp(-kn^2\pi^2 t))$ and as t grows only the small n's will survive and as $t \to \infty$ the mode $n=0$ dominates.

$\to$ SMOOTHING PROPERTY

Exercise for home

Change the BCs to homogeneous Neumann

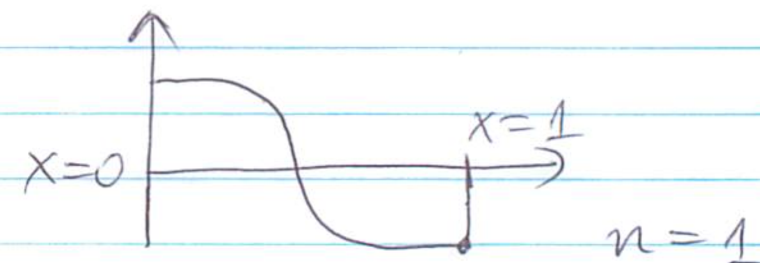
$$u_x(0,t) = u_x(1,t) = 0$$

and show that the separable solution has the form

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-k n^2 \pi^2 t} \cos(\pi n x)$$

This is $n = 0$

satisfies BC



Now the IC becomes the equation for A_0 and A_n:

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x) = g(x)$$

Note that the same derivation can be generalized to the wave equation (and others)