PDE Spring 2016

Lecture 1 & 2

ODE: $$\frac{du(t)}{dt} = f(t, u(t))$$

or $$F\left(t, u(t), \frac{du}{dt}, \frac{d^2u}{dt^2}, \dots\right) = 0$$

For PDEs $$u = u(\underbrace{x, y, z}_{\text{"space"}}, \dots, \underset{\uparrow \atop \text{"time"}}{t})$$

Notation:

$$\frac{\partial u}{\partial x} \text{ or } u_x \text{ or } \partial_x u$$

$$\frac{\partial^2 u}{\partial t \, \partial u} \text{ or } u_{tx} \text{ or } \partial_{tx} u$$

$\uparrow \parallel$

u_{xt} when twice cont diff.

$$\frac{\partial^2 u}{\partial x^2} \text{ or } u_{xx} \text{ or } \partial_x^2 u$$

PDE order

The order of the PDE is the order of the highest derivative that appears in

$$F(u, x, y, z, \dots, t, u_x, u_y, u_z, \dots, u_t, u_{xx}, u_{xy}, \dots, u_{tt}, u_{tx}, \dots) = 0$$

Solving the PDE means to find a sufficiently differentiable function that satisfies the equation identically.

e.g. $u_t = u_{xx}$ → second order

Try $u(x,t) = e^{-t} \sin x$

$$u_t = -e^{-t}\sin x = -u$$

$$u_x = e^{-t}\cos x, \quad u_{xx} = -e^{-t}\sin x = -u$$

$\Rightarrow u_t = u_{xx}$ indeed

Example PDEs

③

First order

Advection or one-way wave eq:

① $u_t + c u_x = 0$ (linear)

Inviscid Burgers

② $u_t + u u_x = 0$ (non-linear)

Second-order

③ Laplace's eq. (linear)

$u_{xx} + u_{yy} = 0$ in 2D

$\underbrace{u_{xx} + u_{yy} + u_{zz}}$ $= 0$ in 3D

$\nabla^2 u = \Delta u = 0$ in any d

Laplacian

$$\sum_{i=1}^{N_{vars}} \frac{\partial^2 u}{\partial x_i^2} = 0$$

④ Poisson's equation (linear)

$$\nabla^2 u = f(x, y, z, \ldots)$$

5 Heat or diffusion eq.

$$u_t = k u_{xx} \quad \text{(linear)}$$

or $u_t = k \nabla^2 u$

6 Wave eq. (two-way)

$$u_{tt} = c^2 u_{xx} \quad \text{(linear)}$$

$$u_{tt} = c^2 \nabla^2 u$$

→ SKIP

9 Black-Scholes Eq. (linear)

~~advection diffusion eq.~~

$$u_t + \frac{\sigma^2}{2} x^2 u_{xx} + r x u_x = r u$$

option price → u

volatility → σ

$x \equiv$ stock price

← RETURN LATER

7 Advection-diffusion-reaction eq.

$$u_t = k \nabla^2 u + r u \quad \text{(linear)}$$

↑ reaction rate

e.g. population dynamics

8 Viscous Burgers (non-linear)

$$u_t + u u_x = k u_{xx}$$

RETURN

Third order

(10) Korteweg-de Vries (KdV) (5)

$$u_t + 6uu_x + u_{xxx} = 0$$

soliton waves

Example solutions

(1) $u_t = u_{xx}$

$$u = \frac{1}{\sqrt{t}} e^{-x^2/4t}$$

Exercise : confirm math vs (Discuss physics

Dimensional analysis

$x \equiv [m]$ length

$u_t = k\, u_{xx}$ $\quad t = [s]$ time

time $\quad$ space

$\frac{\partial}{\partial x}$ has units of $[m^{-1}]$ inv length

$\frac{\partial}{\partial t}$ has units $[s^{-1}]$ inv time

u has units $[u = ?]$

$$u_t \equiv \left[\frac{u}{s}\right] = k \cdot u_{xx} = k \left[\frac{u}{m^2}\right]$$

Therefore

$$k \equiv \left[\frac{m^2}{s} \right] \equiv \text{diffusion coeff.}$$

. Take

$$u_t = u_{xx}, \quad u = \frac{1}{\sqrt{t}} e^{-x^2/4t}$$

$$u_t = k u_{xx}$$

$$u_t = \left[\frac{1}{k \cdot \text{time}} \right] = u_{xx} = \left[\frac{1}{m^2} \right]$$

Changing k is the same as changing the <u>unit</u> of time

Only the product kt can appear in the solution.

$$u = \frac{1}{\sqrt{kt}} e^{-\underbrace{x^2/(4kt)}_{\text{dimensionless}}}$$

$$\text{Typical time} = \frac{\text{Length}^2}{k} \equiv [S]$$

$\dfrac{x^2}{kt}$ enters in the solution together

But our formula still has units of 7

$$\frac{1}{\sqrt{kt}} = \left[\frac{1}{\sqrt{\frac{m^2}{s}\cdot s}}\right] = \frac{1}{m}$$

and not units of $[u]$.

So this solution cannot be the physically-relevant one

Instead, let's take

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-s)^2/(4kt)} \cdot g(s)\, ds$$

where $g(s)$ is a function and g has units $[u]$

→ This has proper units now!

But is it a solution?

$$u_t = -\frac{1}{2t\sqrt{4\pi k t}} \left(\int \right) + \frac{1}{\sqrt{4\pi k t}} \int \frac{(x-s)^2}{4kt^2} e^{-(x-s)^2/(4kt)} g(s)\, ds$$

$$\frac{1}{\sqrt{4\pi k t}} \left[-\frac{1}{2t} \left(\int \right) + \frac{1}{4kt^2} \left(\int (x-s)^2 \dots \right) \right]$$

Practice confirm units are ok

$$u_x = \dots$$

finish as exercise

There is an easier way to prove this as we will see later

Why did we choose $\frac{1}{\sqrt{4\pi}}$?

We will show later that

$$\lim_{t \to 0} u(x,t) = g(x)$$

with this normalization

the function $\frac{1}{\sqrt{4\pi t}} e^{-t^2/4t}$

is called the <u>Green's function</u>

for the heat/diff eq. in 1D. We

will discuss this much later in the

class.

Practice:

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Show that $u = \ln r$

(a) $r = \sqrt{x^2+y^2}$ satisfies Laplace's $\nabla^2 u = 0$ in 2D

(b) $u = \frac{1}{r}$ solves $\nabla^2 u = 0$ in 3D

(these are related to Green's functions)