Lyapunov Functions

Consider the autonomous system

\[ \dot{y} = \dot{\mathbf{z}} \]

where \( \dot{\mathbf{z}} \) and \( \dot{\mathbf{y}} \) are continuous.
Assume that \( \dot{y} = 0 \) is the only (isolated) critical point

\[ \dot{\mathbf{y}}(0) = 0 \] (fixed point)

In the non-linear setting we cannot apply any of our previous theorems.

There are some general statements we can make however:

1. At most a single trajectory (orbit) passes through any non-critical point.

2. If an orbit passes through a non-critical point, it will take it an infinite time to reach a critical point.
(3) An orbit (trajectory) that passes through at least one non-critical point cannot cross itself (but it can be a closed curve).

These are qualitative statements that do not say much about stability.

To make precise statements, sometimes we can use an approach first devised by Lyapunov.

Definition: A scalar function

\[ V(y) : \mathbb{R}^n \rightarrow \mathbb{R} \]

is positive definite if

\[ V(\vec{0}) = 0 \quad \text{and} \quad V(\vec{y}) > 0 \quad \text{for any non-zero} \ \vec{y}. \]

If \( V(\vec{y}) < 0 \) for all \( \vec{y} \neq \vec{0} \) it is negative definite.
Take a solution $y(t)$ of the ODE

$$ y' = f(y) $$

and look at the value

$$ V(y(t)) = \varphi(t) $$

as a function of $t$.

The rate of change of $\varphi(t)$ is

$$ \frac{d\varphi}{dt} = \frac{dV(y(t))}{dt} = \frac{\partial V}{\partial y_1} y_1' + \frac{\partial V}{\partial y_2} y_2' + \cdots + \frac{\partial V}{\partial y_n} y_n' = \nabla V \cdot \vec{y}' = \nabla V \cdot (\vec{\dot{y}}(y)) $$

If $\frac{d\varphi}{dt} \geq 0$ that means $V(y(t))$ is a non-increasing function of time.
Example

\[
\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -y_1 - 2y_2
\end{align*}
\] (linear)

Define \( V(y_1, y_2) = \frac{1}{2} (y_1^2 + y_2^2) \)
which is positive-definite (why?)

Then

\[
\frac{dV(y(\cdot))}{dt} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \cdot \begin{bmatrix} y_2 \\ -y_1 - 2y_2 \end{bmatrix} =
\]

\[= y_1 y_2 - y_2 y_1 - 2y_2^2 = -2y_2^2 \leq 0 \]

so \( V(y(\cdot)) \) does not increase with time.

Let us draw a contour plot of \( V(y_1, y_2) \) by looking at the contour

\[
V(y_1, y_2) = \frac{1}{2} (y_1^2 + y_2^2) = c > 0
\]

\[\Rightarrow y_1^2 + y_2^2 = 2c = r^2 \]
where $r = \sqrt{2}c$ is the radius of the circular contour.

The region $V(y_1, y_2) \leq 1$ is a disk of radius $\sqrt{2}$.

If we have an initial condition $y_1 = (y_0)_1$ and $y_2 = (y_0)_2$ then the initial value of $V(0) = \frac{1}{2} [(y_0)_1^2 + (y_0)_2^2]$ and since $V(t)$ does not increase the trajectory has to remain inside the disk

$$y_1^2 + y_2^2 \leq (y_0)_1^2 + (y_0)_2^2$$
So the point $y=0$ is stable!

This is a rather general result and can be made into Lyapunov's theorem:

Theorem: if there exists a $V(y)$ that is positive definite and if

$$\frac{dV(y(t))}{dt} \leq 0$$

then $y=0$ is a stable point.

If $\frac{dV}{dt} < 0$ then it is an asymptotically stable point.
Example: Pendulum (undamped by friction)

\[ \begin{align*}
\frac{dy_1}{d\tau} &= y_2 \\
\frac{dy_2}{d\tau} &= -y_2 - w^2 \sin y_1,
\end{align*} \]

where \( w > 0 \) is the swinging frequency.

\[ V(y_1, y_2) = \frac{1}{2} y_2^2 + w^2 (1 - \cos y_1) \]

(follows from physics as the total energy of the pendulum, sum of kinetic and potential energy)

Show that (DIY)

\[ \frac{dh}{d\tau} = \frac{d}{d\tau} V(y(t)) = 0 \]

This means that the orbits/trajectories of the system are the contours of \( V(y) \):

\[ \frac{1}{2} y_2^2 + w^2 (1 - \cos y_1) = h \]
\[ y_2 = \pm \sqrt{2 \left[ h - w^2 (a - \cos y) \right]} \]

So \( y = 0 \) is a stable point (not asymptotically stable) if \( 0 < h \leq 2w^2 \).

But if \( h > 2w^2 \), then \( y_2 \) never reaches zero (you push the pendulum too hard/fast).