A. Donev
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Lyapunov Functions

Consider the autonomous system

$$\vec{y}\,' = \vec{f}(\vec{y})$$

where $\vec{f}$ and $\vec{\nabla}\vec{f}$ are continuous.

Assume that $\vec{y} = \vec{0}$ is the only (isolated) critical point

$$\vec{f}(\vec{0}) = \vec{0} \quad \text{(fixed point)}$$

In the non-linear setting we cannot apply any of our previous theorems.

There are some general statements we can make however:

1. At most a single trajectory (orbit) passes through any non-critical point

2. If an orbit passes through a non-critical point, it will take it an infinite time to reach a critical point.

(3) An orbit (trajectory) that passes through at least one non-critical point cannot cross itself (but it can be a closed curve)

These are qualitative statements that do not say much about stability.

To make precise statements, sometimes we can use an approach first devised by Lyapunov.

<u>Definition</u>: A scalar function

$$V(y): \mathbb{R}^n \to \mathbb{R}$$

is <u>positive definite</u> <u>iff</u> $V(\vec{0}) = \vec{0}$ and $V(\vec{y}) > 0$ for any non-zero $\vec{y}$.

If $V(\vec{y}) < 0$ for all $\vec{y} \neq 0$ it is <u>negative-definite</u>

Take a solution $y(t)$ of the ODE

$$y' = f(y)$$

and look at the value

$$V(y(t)) = v(t)$$

as a function of t.

The rate of change of $v(t)$ is

$$\frac{dv}{dt} = \frac{d\,V(y(t))}{dt} = \frac{\partial V}{\partial y_1} \cdot y_1' + \frac{\partial V}{\partial y_2} \cdot y_2' + \dots + \frac{\partial V}{\partial y_n} \cdot y_n' =$$

$$= (\vec{\nabla} V) \cdot \vec{y}\,' = (\vec{\nabla} V) \cdot (\vec{f}(\vec{y}))$$

If $\frac{dv}{dt} \geq 0$ that means $V(y(t))$ is a non-increasing function of time

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Example

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 - 2y_2 \end{cases} \quad \text{(linear)}$$

Define $V(y_1, y_2) = \frac{1}{2}(y_1^2 + y_2^2)$

which is positive-definite (why?)

then

$$\frac{dV(y(t))}{dt} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \cdot \begin{bmatrix} y_2 \\ -y_1 - 2y_2 \end{bmatrix} =$$

$$= y_1 y_2 - y_2 y_1 - 2y_2^2 = -2y_2^2 \leq 0$$

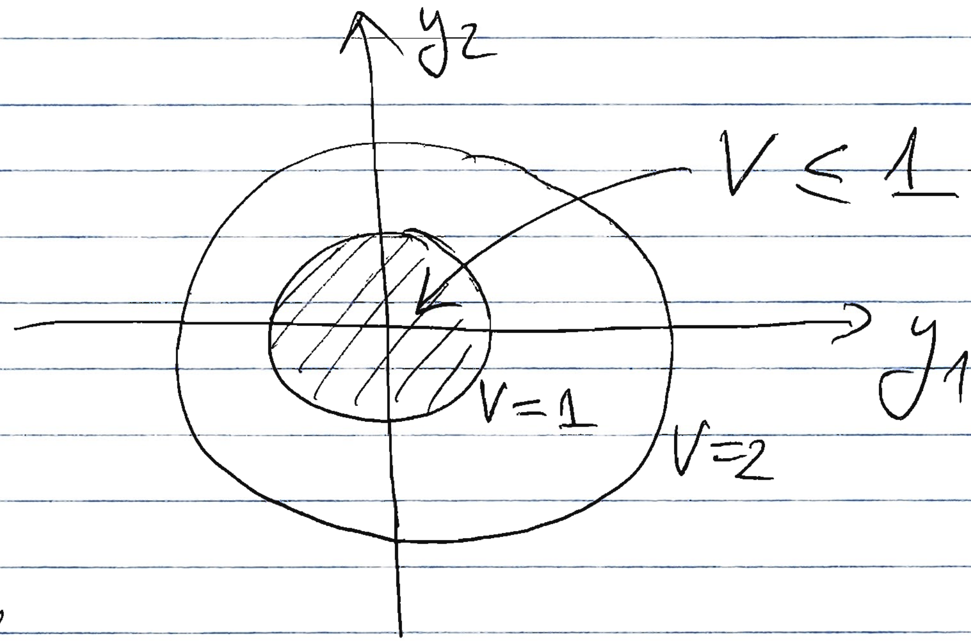
So $V(y(t))$ does not increase with time.

Let us draw a contour plot of $V(y_1, y_2)$ by looking at the contour

$$V(y_1, y_2) = \frac{1}{2}(y_1^2 + y_2^2) = c > 0$$

$$\Rightarrow \quad y_1^2 + y_2^2 = 2c = r^2$$

where $r = \sqrt{2c}$ is the radius of the circular contour



The region $V(y_1, y_2) \leq 1$ is a disk of radius $\sqrt{2}$.
If we have an initial condition $y_1 = (y_0)_1$ and $y_2 = (y_0)_2$ then the initial value of $v(0) = \frac{1}{2}\left[(y_0)_1^2 + (y_0)_2^2\right]$ and since $v(t)$ does not increase the trajectory has to remain inside the disk

$$y_1^2 + y_2^2 \leq (y_0)_1^2 + (y_0)_2^2$$

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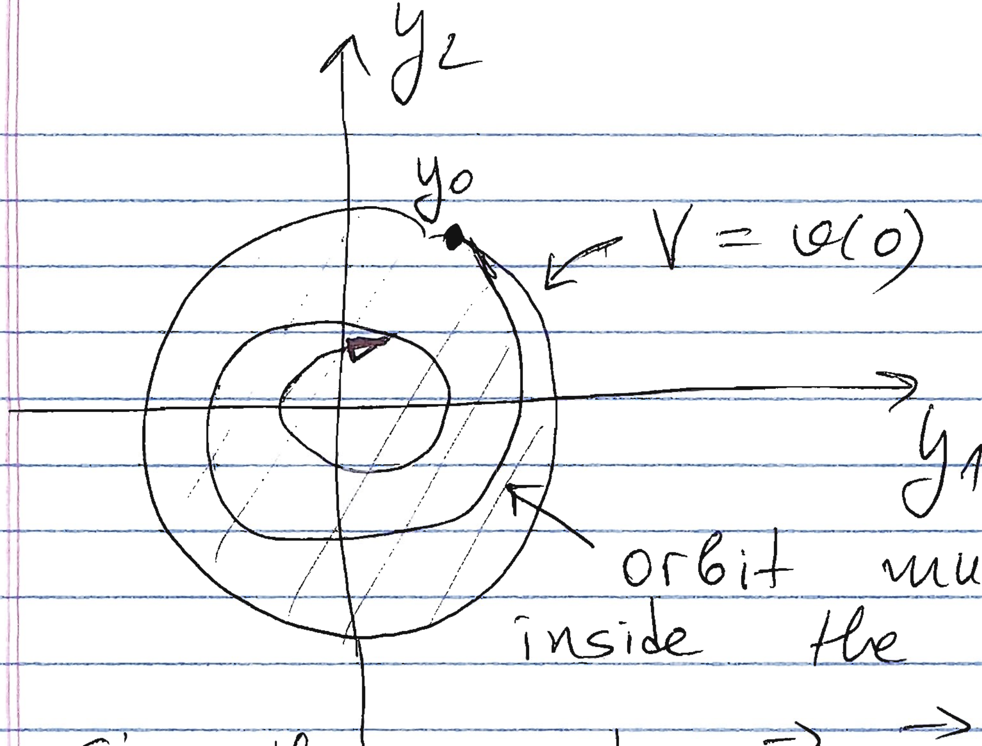


So the point $\vec{y} = \vec{0}$ is <u>stable</u> !

This is a rather general result and can be made into Lyapunov's theorem:

<u>Theorem:</u> If there exists a $V(\vec{y})$ that is positive-definite and if

$$\frac{d\,V(\vec{y}(t))}{dt} \leq 0$$

then $\vec{y} = \vec{0}$ is a <u>stable point</u>.

If $\frac{dV}{dt} < 0$ then it is an <u>asymptotically stable</u> point.

Example: Pendulum (undamped by friction)

$$\begin{cases} y_1' = y_2 \\ y_2' = \cancel{-\frac{g}{l}\sin y_1} \; -w^2 \sin y_1 \end{cases}$$

where $w > 0$ is the swinging frequency.

$$V(y_1, y_2) = \frac{1}{2} y_2^2 + w^2(1 - \cos y_1)$$

(follows from physics as the total energy of the pendulum, sum of kinetic and potential energy)

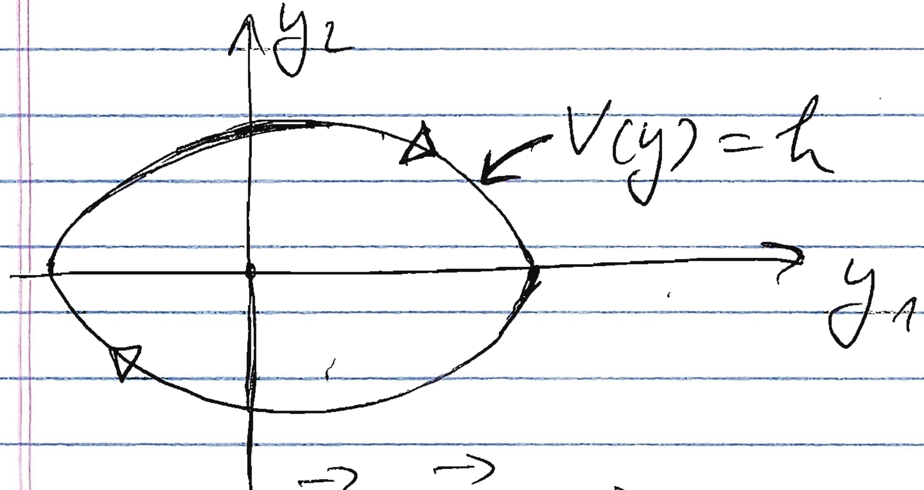
Show that (DIY)

$$\frac{dv}{dt} = \frac{d\,V(y(t))}{dt} = 0$$

This means that the orbits/trajectories of the system are the contours of $V(y)$:

$$\frac{1}{2} y_2^2 + w^2(1 - \cos y_1) = h$$

$$\Rightarrow \quad y_2 = \pm\sqrt{2\left[h - w^2(1-\cos y_1)\right]}$$



So $\vec{y} = \vec{0}$ is a stable point (not asymptotically stable) if $0 < h \leq 2w^2$

But if $h > 2w^2$, then y_2 never reaches zero (you push the pendulum too hard/fast)