1. **Constant Coefficients**

   **General Theory**

   \[ \text{alg} \geq \text{geometric multiplicity} \]

   Assume that \( A \) is diagonalizable (always the case if eigenvalues are distinct).

   \[ A = PDP^{-1} \]

   \[ P = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix} \]

   \( \uparrow \) linearly independent eigenvectors as columns

   \[ D = \begin{bmatrix} \lambda_1 & & \ & \ \ \\\n                     & \lambda_2 & & \ \\
                     & & \ddots & \ \\
                     & & & \lambda_n \end{bmatrix} \]

   diagonal matrix of eigenvalues

   \[ y' = Ay = PD P^{-1} y \]

   Introduce a new set of variables (new basis)

   \[ z = P^{-1} y \quad \Rightarrow \quad z' = P'^{-1} y' \]

   \[ z' = P'^{-1} y' = (P'^{-1} P) D P^{-1} y = D z \]
In the new basis
\[ z' = Dz \]
which means the linear transformation \( P \) diagonalizes the system of equations.

Each \( z_k \) is independent:
\[ z'_k = \lambda_k z_k \Rightarrow z_k = c e^{\lambda_k t} \]

\[ y = Pz = \sum_{k=1}^{n} z_k \chi_k \]
\[ y = \sum_{k=1}^{n} c_k e^{\lambda_k t} \chi_k \]

which is exactly what we obtained earlier.

\[ \begin{cases} \text{Note: If } D \text{ were lower or upper triangular then one could solve } \quad z' = Dz \quad \text{by forward / backward substitution} \end{cases} \]
What if $A$ is not diagonalizable?

$A = T \begin{bmatrix} \lambda_1 & 1 \\ & \ddots & \ddots \\ & & \ddots & 1 \\ & & & \lambda_n \end{bmatrix} T^{-1}$

for any matrix, where $J$ is the Jordan canonical form.

$J = \begin{bmatrix} \lambda_1 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & \ddots & & & & \\ & & & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & \ddots \\ & & & & & & & \lambda_4 \end{bmatrix}$

The number of Jordan blocks is geometric multiplicity of eigenvalue (one linearly independent eigenvector per block).

Jordan form is hard to find in general, must do by hand.
\[ z = T^{-1} y \]
\[ z' =\]

Notice that each Jordan block is decoupled (independent) from the other blocks.

So consider solving one block of size 2:

\[ z' = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} z = A z \]

Claim:
\[ \phi = e^{xt} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \]

is a fundamental matrix.

Check:
\[ y_2 = \begin{bmatrix} te^{xt} \\ e^{xt} \end{bmatrix} \]
\[ y' = \begin{bmatrix} e^{xt} + \lambda te^{xt} \\ \lambda e^{xt} \end{bmatrix} = A y_2 \Rightarrow y_2 \text{ is a solution} \]
How could we obtain this ourselves?

\[ z' = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} z \]

\[
\begin{cases}
    z'_1 = \lambda z_1 + z_2 \\
    z'_2 = \lambda z_2
\end{cases}
\]

Solve for \( z_2 \) first, \( z_2 = ce^\lambda t \)

\[ z'_1 = \lambda z_1 + ce^\lambda t < \text{non-homogeneous} \]

\[
\begin{cases}
    z_1 = c_2 e^\lambda t + c_1 t e^\lambda t \\
    z_2 = ce^\lambda t
\end{cases}
\]

which is exactly \( z = \phi c \)

where we already obtained \( \phi \)

and \( c = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} \) in this notation.
How do we obtain $T$ if we know $J$?

$$A = T J T^{-1}$$

$$AT = TJ \leftarrow \text{system of } \frac{n^2}{n} \text{ linear equations for } T_{ij}, i = 1, \ldots, n$$

There are better ways but we won't really use Jordan form to solve linear systems in practice.

Note: One of the columns of $T$ will be the eigenvector but doing the above procedure is the same and gives you the whole matrix.

The full (complicated!) procedure for obtaining $T$ & $J$ is in the book by Miller & Michel, bottom of page 84.
Elimination of variables

\[
\begin{align*}
    \tau_1 &= a_{11} \tau_1 + a_{12} \tau_2 \\
    \tau'_2 &= a_{21} \tau_1 + a_{22} \tau_2 \\
    \tau &= A \tau \\
    a_{21} &\neq 0
\end{align*}
\]

Define new variable

\[
\tau_3 = \tau_1 + \lambda \tau_2 \quad \text{such that}
\]

\[
\tau'_3 = \tau'_1 + \lambda \tau'_2 = \beta \tau_3
\]

\[
\begin{align*}
    &= a_{11} \tau_1 + \lambda a_{21} \tau_1 \\
    &\quad + a_{12} \tau_2 + \lambda a_{22} \tau_2 \\
    &= \beta \tau_1 + \lambda \beta \tau_2
\end{align*}
\]

Equating coefficients in front of \( \tau_1 \) and \( \tau_2 \)

\[
\begin{align*}
    a_{11} + \lambda a_{21} &= \beta \\
    a_{12} + \lambda a_{22} &= \lambda \beta
\end{align*}
\]

Eliminate \( \beta \) to get a quadratic equation for \( \lambda \)

Don't remember by heart!

There could be two solutions for \( \lambda \) or one.
Turns out

\[ \lambda = \frac{\chi - \alpha_{11}}{\alpha_{21}} \quad \beta = 2 \]

where \( \lambda \) is an eigenvalue of the matrix \( A \), which shows the relation with previous methods.

Example

\[ A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \]

Solve \( y' = Ay \)

\[ |A - \lambda I| = \lambda^2 - 6\lambda + 9 = 0 \]

\( \Rightarrow \lambda = 3 = \lambda_2 \)

\[ \begin{bmatrix} A - \lambda I & 0 \\ -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \]

\( \Rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \leftarrow \text{two free parameters} \)
There is only one independent eigenvector

\[ x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

and not a second one.

The Jordan form is

\[ J = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \]

\[ T = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \]

DIY: Obtain \( T \) from \( J \)

An alternative is to do the elimination and figure out

\[ \begin{align*}
\beta &= \lambda = 3 \\
\alpha &= \frac{\lambda - a_{11}}{a_{21}} = \frac{3 - 2}{-1} = 1
\end{align*} \]
\[
\begin{align*}
\tau_3' &= \beta \tau_3 \quad \Rightarrow \quad \tau_3 = c_1 e^{\beta t} \\
\tau_3 &= c_1 e^{3t} \\
\Rightarrow \quad \tau_3 &= c_1 e^{3t} = \tau_1 + \alpha \tau_2 = \\
&= \tau_1 - \tau_2 \\
\tau_1 &= \tau_2 + c_1 e^{3t} \\
\intertext{Recall one of the equations, say} \\
\tau' &= A \tau \\
\begin{cases}
\tau_1' &= 2 \tau_1 + \tau_2 = 3 \tau_2 + 2 c_1 e^{3t} \\
\tau_2' &= -\tau_1 + 4 \tau_2 = 3 \tau_2 - c_1 e^{3t}
\end{cases}
\intertext{Try to solve the same system using the Jordan form of } A \quad \text{(DIY)} \quad \text{and verify you get the same solution}