Convergence: Spatial Discretization

Stability + Consistency = Convergence

\[ U_t(x,t) = f(U(x,t), x, t) \]

\[ W'(t) = F_h(t, W(t)) \iff \text{semi-discrete} \]

\[ W(t) \in \mathbb{R}^m \]

Spatial error:

\[ E(t) = U_h(t) - W(t) \]

where \( U_h(t) \leftrightarrow U(t) \) on h-grid

(e.g., finite difference or finite-volume)

Global error:

\[ \| E(t) \| = O(h^p) \quad \text{for} \quad 0 < t \leq T \]

p-order of convergence
Local error

\[ e_h(t) = u_h(t) - F(t, u_h(t)) \]

Truncation analysis usually done using Taylor series assuming sufficient smoothness

\[ \| e_h(t) \| = O(h^q) \], \quad q > 0 \text{ (usually integer)}

\[ q = \text{consistency order} \]

Usually \( p = q \) but not always (see upcoming lecture on BCs)

\[ p \geq q \]
Linear PDE + linear scheme:

\[ w'(t) = A w(t) + g(t) \quad \ldots \quad (*) \]

\[ A \in \mathbb{C}^{m \times m} \text{ is constant} \]

\[ (\ast) \text{ is stable if } \begin{cases} \| e^{tA} \| \leq Ke^{\omega t} \text{ for } 0 \leq t \leq T \\ K \geq 1 \text{ and } w \in \mathbb{R} \text{ independent of } h \text{ [uniform in } h] \end{cases} \]

\[ \implies \text{ bound on global error if consistent. This is a precise version} \]

\[ \text{(but not general) of } \]

\[ \text{consistency + stability } \implies \text{ convergence} \]
\( \varepsilon'(t) = u'(h) - w'(t) \)

\[ = (s + F(u_h)) - F(w) = A (u_h - w) + \delta_h = A \varepsilon + \delta_h \]

\[ \Rightarrow \varepsilon(t) = e^{tA} \varepsilon(0) + \int_0^t e^{(t-s)A} \delta_h(s) \, ds \]

\[ \| \varepsilon \| \leq \| e^{tA} \| \| \varepsilon(0) \| + \int_0^t \| e^{(t-s)A} \| \| \delta_h(s) \| \, ds \leq \max \| \delta_h \| \][\text{max } \| \delta_h \| \leq C] \]

\[ \| \varepsilon \| \leq e_0 KE + Ke \int_0^t e^{w(t-s)} \, ds \cdot \max \| \delta_h(t) \| \]

Assume consistency of order \( q \):

\[ \| \delta_h(t) \| \leq C h^q \]

\[ \| \varepsilon \| \leq e_0 h^q \text{ (initial truncation error) } \]
If $w \geq 0$, linear or exponential growth of errors in time.

If $w < 0$, then we can bound the error for all $T \geq 0$.

$w < 0 \Rightarrow$ global error $\leq$ local error.

The difficulty is finding a sharp estimate for $w$, i.e., smallest possible $w$.

Normal $A$:

$AA^* = A^*A$  $A = U \Lambda U^{-1}$

$\Rightarrow e^{tA} = U e^{t\Lambda} U^{-1}$
Aside on norms and things

Normal $A$: $\| e^{tA} \| \leq \text{cond}(U) \cdot \max_k |e^{\lambda_k t}|$

condition number = $\|U\| \cdot \|U^{-1}\|$

$\Rightarrow \quad \omega = \max_k \text{Re}(\lambda_k)$

For non-normal $A$ one can use logarithmic "norm"

$\omega = M(A) = \lim_{t \to 0} \frac{\|I + tA\| - 1}{t}$

\[
\begin{cases}
    M_1(A) = \max_j \text{Re} a_{jj} + \sum_{i \neq j} |a_{ij}| \\
    M_\infty(A) = \max_i \text{Re} a_{ii} + \sum_{i \neq j} |a_{ij}|
\end{cases}
\]
Note that vector (error) norms here should have grid-size (cell-volume) in them:

\[
\begin{align*}
\| \mathbf{v} \|_1 &= h \sum_j |v_j| 
&\approx \int |u(x)| \, dx \\
\| \mathbf{v} \|_2 &= h \sum_j u_j^2 
&\approx \int |u(x)|^2 \, dx \\
\| \mathbf{v} \|_\infty &= \max_j |v_j| 
&\approx \max_j |u(x)|
\end{align*}
\]
Example: Adv-diff equation:

$$u_t + au_x = d u_{xx} \quad \text{periodic}$$

$$\mathbf{E}(t) = \sum_k \mathbf{E}_k(t) \chi_k \quad \rightarrow \text{Fourier basis}$$

$$\mathbf{G}_k(t) = \sum_l \mathbf{G}_{k,l}(t) \chi_l$$

$$\Rightarrow \mathbf{\hat{E}}_k = \lambda_k \mathbf{\hat{E}}_k(t) + \mathbf{\hat{G}}_{k,l}(t) \quad \rightarrow \text{scalar ODE}$$

For linear equations, different (discrete) wavenumbers are fully decoupled.

$$\Rightarrow \omega = \max_{k \neq 0} \text{Re} \left( \omega_k \right) \leq \text{exclude } k = 0 \quad \text{due to conservation}$$
Going back to our eigenvalue calculation:

\[
\begin{aligned}
    w &= \begin{cases} 
        -\frac{4d}{h^2} \sin^2(\pi x) \geq -4d \frac{1}{h^2} \\
        -2 \left| \frac{a}{h} \right| \sin^2(\pi x) \geq -2 \left| \frac{a}{h} \right| \frac{1}{h^2} 
    \end{cases}
\end{aligned}
\]

Centered 2nd order centered 2nd order

\[\text{first-order upwmd}\]

not uniform in \( h \)

but useful for estimates

\[\text{Homework: } U_t + U_x = 0, \quad U(x, 0) = (\sin(\pi x))^{100}_{0 \leq x \leq 1}\]

Plot evolution of \( \| E(t) \|_2 \) for

\[m = 100, 200, 400, 800 \] for \( 0 \leq t \leq 10 \)

Make error relative to something

Compare to theory
Variable Coefficients

\[ u_t + \left[ a(x) u \right]_x = \left[ d(x) u_x \right]_x \]

Conservation law \rightarrow \text{advection flux} \rightarrow \text{diffusive flux}

Recall \[ \overline{u}_j(t) = w_j = \frac{1}{h} \int_{x_{j-h/2}}^{x_{j+h/2}} u(x, t) \, dx \]

Tayler series \[ = u(x_j, t) + \frac{h^2}{24} u_{xx}(x_j) + \ldots \]

We can choose finite-difference

\[ \begin{cases} w_j = u(x_j, t) \quad \text{or finite-volume} \\ w_j = \overline{u}_j(t) \end{cases} \]

\( \Rightarrow \) Makes no difference up to second order
But for discretization, we use finite-volume, which means we use flux form or conservation form:

$$w_j(t) = \frac{1}{h} \left[ F_{j-1/2} - F_{j+1/2} \right]$$

Fluxes through cell boundaries

Let's take the natural

$$F_{j+1/2} = a(x_{j+1/2}) w_{j+1/2} - d(x_{j+1/2})(w_j - w_{j+1})$$

Advection flux
Diffusive flux

What is $w_{j+1/2}$?
How to go from cell centers to cell faces?
If \( a(x) > 0 \) for all \( x \)

\[
\omega_{j+1/2} = \omega_j
\]

is \textit{upwind (first-order)}

more generally, one writes

\[
\begin{aligned}
\int a(x_{j+1/2}) \omega_{j+1/2} &= a^+ (x_{j+1/2}) \omega_j(t) - a^- (x_{j+1/2}) \omega_{j+1} \\
\text{where } a^+ &= \max(a, 0), \quad a^- = \min(a, 0)
\end{aligned}
\]

for \textit{first-order upwind scheme}

or

\[
\begin{aligned}
\omega_{j+1/2} &= \frac{1}{2} (\omega_j + \omega_{j+1}) \\
\text{for second-order centered scheme}
\end{aligned}
\]

\textbf{DIY:} Convince yourself that for \textit{Do-It-Yourself constant coeff. these are the same as before.
One can prove stability for these by using, for example, the logarithmic "norm" $\| M(A) \|$. For pure advection, one gets:

$$M_\infty(A) = \omega = \frac{1}{k} \max_j \left[ a(x_{j-1/2}) - a(x_{j+1/2}) \right] = O(1)$$

and for pure diffusion:

$$M_1(A) \leq 0 \quad \text{and} \quad M_\infty(A) \leq 0 \Rightarrow \| e^{tA} \|_1 \leq 1 \quad \text{and} \quad \| e^{tA} \|_\infty \leq 1$$

and the same holds for $L_2$ norm.
For third-order upwind-biased:

\[ W_{j \pm 1/2} = \begin{cases} \frac{1}{6} [-W_{j-1} + 5W_j + 2W_{j+1}] \\ \text{if } a(x_{j+1/2}) \geq 0, \text{ and similarly} \\ \text{flip direction for } a_{j+1/2} \leq 0 \end{cases} \]

Analytical exercise:

Show that the order of consistency of this scheme is \( q = 2 \) for finite-difference interpretation, but \( q = 3 \) for finite-volume.

No stability result exists in general—

we use heuristics and frozen coefficients arguments if \( a \) is smooth.