Incompressible Flow

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\[ \begin{align*}
\vec{u}_t + \nabla \cdot \vec{u} &= \alpha \left( \vec{u} \cdot \nabla \right) \vec{u} + \nu \nabla^2 \vec{u} + \text{other} \\
\text{Lagrange multiplier} &\quad \text{advection} & \nu \quad \text{kinematic viscosity}
\end{align*} \]

\[ \nabla \cdot \vec{u} = 0 \quad \text{(incompressibility)} \]

Note

\[ \vec{u} \cdot \nabla \vec{u} \equiv \nabla \cdot (\nu \vec{u}) \]

\[ u_i \rightarrow u_j \quad u_i \vec{e}_j u_i \equiv \partial_j (u_i \vec{e}_j u_i) \]

Since

\[ u_j \vec{e}_j u_i + u_i \vec{e}_j u_j \]

\[ \begin{align*}
\tau_{i} + \tau_{i} p &= -\eta_{j} \tau_{j} + \nu \tau_{j} \tau_{j} \\
\tau_{j} \tau_{j} &= 0, \quad i = 1, 2, \ldots, d \quad (d=2 \text{ or } d=3) 
\end{align*} \]

Implied summation (Einstein) convention for repeated indices.

This form of the equations applies only if density is constant.

\[ \begin{align*}
\rho &= \text{const.} \\
\nu &= \frac{\eta}{\rho} < \text{viscosity}
\end{align*} \]
Otherwise, one needs to solve

\[
\left\{\begin{array}{l}
\left( \sigma \dot{u} \right)_t + \nabla p = - \nabla \cdot (\sigma \dot{u} \otimes \dot{u}) + \nabla \cdot \left( \sigma \right) \\
\text{momentum conservation}
\end{array}\right.
\]

\[S_t + \dot{u} \cdot \nabla \sigma = 0 \leq \text{continuity equation (conservation of mass)}
\]

\[\nabla \cdot \dot{u} = 0 \]

Equivalent formulation:

\[\sigma \dot{u}_t + \sigma \dot{u} \cdot \nabla \dot{u} = \nabla \cdot \left( \sigma \right) + \rho g \]

\[\text{advection stress tensor}\]
Here the stress tensor
\[ \sigma = -p \mathbf{I} + \eta (\mathbf{D} \mathbf{u} + \mathbf{D}^T \mathbf{u}) \]

- Mechanical stress
- Viscous stress (tensor)

Note that if \( \eta = \text{const} \)

\[ \sigma_{ij} = -p \delta_{ij} + \eta (\partial_i u_j + \partial_j u_i) = -\partial_i p + \eta (\partial_j^2 u_i + \partial_i \partial_j u_j) = -\partial_i p + \eta \partial_j^2 u_i + \eta \partial_i (\partial_j u_j) \]

\[ \Rightarrow \Delta \mathbf{u} = -\nabla p + \nabla \cdot \mathbf{D}^2 \mathbf{u} = \text{zero} \]
To summarize variable-density variable-viscosity equations:

\[
\begin{aligned}
    u_t + u \cdot \nabla u &= -\frac{1}{\rho} \nabla p + \nabla \cdot \left[ \eta \left( \nabla u + (\nabla u)^T \right) \right] \\
    s_t + u \cdot \nabla s &= 0 \\
    \nabla \cdot u &= 0
\end{aligned}
\]

But for now let's focus on the constant-coefficient case

\( s = \text{const} \quad \eta = \text{const} \quad \gamma = \frac{\eta}{s} \)
\[ \begin{aligned} \nabla \cdot u &= 0 \\ \nabla \cdot (u \cdot \nabla u) &= \gamma \nabla^2 u + \text{forcing} + \ \nabla \cdot (u \cdot \nabla c) &= \chi \nabla^2 c + \text{forcing} \end{aligned} \]

Concentration or density of a passively advected scalar (e.g., a pollutant advected by the flow of air).

As we can see, these are basically advection-diffusion equations with a twist:

\[ \begin{aligned} \Rightarrow \nabla \cdot (u \cdot \nabla u) \text{ is nonlinear} \\ \Rightarrow \text{The equations are constrained } \nabla \cdot u = 0 \\ \Rightarrow \text{Pressure has no evolution law} \end{aligned} \]
Formally, the NS equations are a differential-algebraic system of equations \((\text{DAE})\) of index 2.

Even if they were simple ODEs, they would be non-trivial to integrate in time!

It is possible to formally eliminate the pressure to get the pressure-free formulation:

\[
\mathbf{u}_t = P \left[ -\mathbf{u} \cdot \nabla \mathbf{u} + \nabla^2 \mathbf{u} + f \right]
\]

where \(P\) is an integro-differential projection operator.
Hodge Decomposition (or)

Helmholtz Theorem

Let \( \mathbf{v} \) be a vector field on a bounded domain in \( \mathbb{R}^3 \), smooth.

\[
\mathbf{v} = -\nabla \phi + \nabla \times \mathbf{A}
\]

irrotational part

divergence free part

uniquely

\[
\mathbf{v} = -\nabla \phi + \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0
\]

If \( \mathbf{v} \) decays at infinity or vanishes on boundary of domain, one can write explicitly
where $P$ is a projection operator that takes a vector field and projects it onto the space of divergence-free vector fields.

$$u = P u$$

$L_2$ projection onto $\nabla \cdot u = 0$

$$P u = \frac{1}{4\pi} \nabla \times \int \frac{\nabla' \times u(r')}{|r-r'|} \, dV'$$

$$= u + \frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot u(r')}{|r-r'|} \, dV'$$
\[ \psi(r) = \frac{1}{4\pi} \int \frac{\nabla' \cdot \mathbf{v}(r')}{|r-r'|} \, dv' \]

\[ \mathbf{A}(r) = \frac{1}{4\pi} \int \frac{\nabla' \times \mathbf{v}(r')}{|r-r'|} \, dv' \]

Note that

\[ \nabla \cdot \mathbf{v} = -\nabla^2 \psi \]

Poisson equation for \( \psi \)

and

\[ -\frac{1}{4\pi} \frac{1}{|r-r'|} \]

is the Green's function

for this Poisson equation

\[ U = \psi + \nabla \psi = \psi - \nabla (\nabla^{-2} \psi) \mathbf{v} \]

defines \( P \mathbf{v} \)
Boundary Conditions

Note: Periodic boundaries are not real (physical) boundaries!

At a physical boundary, the following BCs are allowed: (4 types)

- **normal component**
  - normal velocity: \( \mathbf{u} \cdot \mathbf{n} = u_n \)
  - normal stress (traction): \( \mathbf{n} \cdot \mathbf{\sigma} \cdot \mathbf{n} = -p + 2\eta \frac{\partial}{\partial n} (\mathbf{n} \cdot \mathbf{\sigma} \cdot \mathbf{n}) \)

- **tangential component**
  - \( \mathbf{u} - \mathbf{u} \cdot \mathbf{n} \mathbf{n} = \mathbf{t} \)
  - tangential stress

**PLANAR BOUNDARIES** specified
So the four options are:

**A** \[ \begin{aligned}
\vec{n} \cdot \vec{u} &= u_n \\
\vec{n} \left[ \frac{\partial \vec{u}_n}{\partial n} + \frac{\partial u_n}{\partial \vec{r}} \right] &= \vec{f}_n
\end{aligned} \]

(i.e. Dirichlet for \( u \))

normal and tangential

(NO-SLIP BCs)

**B** \[ \begin{aligned}
\vec{n} \cdot \vec{u} &= u_n \\
\eta \left[ \frac{\partial \vec{u}_n}{\partial n} + \frac{\partial u_n}{\partial \vec{r}} \right] &= \vec{f}_n
\end{aligned} \]

normal velocity

tangential stress

(SLIP BCs)

**C** \[ \begin{aligned}
-p + 2\eta \frac{\partial u_n}{\partial n} &= f_n \\
\vec{n} \left[ \frac{\partial \vec{u}_n}{\partial n} + \frac{\partial u_n}{\partial \vec{r}} \right] &= \vec{f}_n
\end{aligned} \]

normal stress

tangential velocity

**D** \[ \begin{aligned}
-p + 2\eta \frac{\partial u_n}{\partial n} &= f_n \\
\eta \left[ \frac{\partial \vec{u}_n}{\partial n} + \frac{\partial u_n}{\partial \vec{r}} \right] &= \vec{f}_n
\end{aligned} \]

Dirichlet for stress
In practice one often wants "outflow" or transparent BCs, but these are not proper physical BCs since usually the physical conditions are unknown (artificial boundaries).

Note: For non-Newtonian flow, Euler equations, one can only specify:

- Normal component: either normal velocity or pressure. Boundary layers will occur when viscosity is weak (recall cell Péclet number).
Vorticity

\[ \omega = \nabla \times \mathbf{u} \Rightarrow \nabla \cdot \omega = 0 \]

\( \omega = 0 \rightarrow \text{potential flow} \)

\[ \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \nabla |\mathbf{u}|^2 - \mathbf{u} \times (\nabla \times \mathbf{u}) \]

(\text{vector identity})

Note \( \nabla \times (\nabla \mathbf{p}) = 0 \)

\[ \partial_t \mathbf{u} + \omega \times \mathbf{u} + \nabla \mathbf{p} = \nu \nabla^2 \mathbf{u} + \mathbf{f} \]

Vorticity formulation \( = -\nu \nabla \times \omega + \mathbf{f} \)

where we used \( \nabla \times (\nabla \times \omega) = \nabla (\nabla \cdot \omega) - \nabla^2 \omega \)
Apply a curl to equation and use
\[ \nabla \times (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{w} \]
to get \textit{vorticity equation}:
\[ \partial_t \mathbf{w} + \mathbf{v} \cdot \nabla \mathbf{w} = \mathbf{w} \cdot \nabla \mathbf{v} + \nu \nabla^2 \mathbf{w} \]

In two dimensions, \( \mathbf{v} = (u, v, 0) \)
\[ \mathbf{w} \cdot \nabla \mathbf{v} = 0 \], so there is no \textit{vorticity}
generation:
\[ \partial_t \mathbf{w} + \mathbf{v} \cdot \nabla \mathbf{w} = \nu \nabla^2 \mathbf{w} \text{ in 2D} \]
Define a stream function $\psi$

$$u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x}$$

i.e. $\mathbf{u} = (\nabla \psi) \times \hat{z}$

to get Poisson equation for $\psi$

$$\Delta^2 \psi = -\mathbf{\omega}$$

$\mathbf{u} \rightarrow \mathbf{\omega} \rightarrow \psi \rightarrow \mathbf{u}$

$\mathbf{\omega} + \mathbf{\omega} \cdot \nabla \mathbf{u} = \nabla \times (\nabla \times \mathbf{\omega})$ \\
$\mathbf{u} = -\nabla \times \left[ \left( \nabla^2 \mathbf{\omega} \right) \times \hat{z} \right]$ \\
Only vorticity appears!