The Taylor vortex
\[
v_x = v_0 - 2e^{-8\pi^2 \mu t / L^2} \cos \left( \frac{2\pi(x - v_0 t)}{L} \right) \sin \left( \frac{2\pi(y - v_0 t)}{L} \right),
\]
(1)
\[
v_y = v_0 + 2e^{-8\pi^2 \mu t / L^2} \sin \left( \frac{2\pi(x - v_0 t)}{L} \right) \cos \left( \frac{2\pi(y - v_0 t)}{L} \right),
\]
(2)
\[
\pi = -e^{-16\pi^2 \mu t / L^2} \left[ \cos \left( \frac{4\pi(x - v_0 t)}{L} \right) + \cos \left( \frac{4\pi(y - v_0 t)}{L} \right) \right].
\]
(3)
is a well-known exact solution of the unforced Navier-Stokes equation
\[
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \pi + \mu \nabla^2 \mathbf{v},
\]
(4)
in two dimensions for a periodic domain of size \(L \times L\). You can set \(v_0 = 1\), \(L = 1\) and \(\mu = 0.05\) for this homework. Note that to capture the non-zero mean of the velocity in the vorticity-stream function formulation you need to keep track of it separately and add it “by hand”,
\[
\mathbf{v} = \nabla^\perp \psi + v_0.
\]

Write a pseudo-spectral solver for the two-dimensional Navier-Stokes equations in the vorticity-stream formulation, as we discussed in class. Choose a temporal integrator and discuss your choice. Solve the NS equations from time \(t = 0\) (using the Taylor vortex as initial condition) to time \(t = 0.25\) and compare the numerical solution to the exact solution. Discuss (both numerically and analytically) the order of accuracy of the scheme you developed, as well as the computational cost. What would be different (better, worse) if you used a finite-volume spatial discretization?

Optional but encouraged: Solve also a forced scalar advection-diffusion equation for the concentration \(c(r, t)\) a passive tracer,
\[
\partial_t c + \mathbf{v} \cdot \nabla c = \mu \nabla^2 c + f(r, t).
\]
(5)
If you set \(f = -\partial_x \pi\) then \(c = v_x\) from the Taylor vortex is a solution of (5). Use this to test your advection-diffusion solver. Then set \(f = 0\) and \(\mu = 0\) and start with the initial condition
\[
c(x, y; t = 0) = \left[ \sin \left( \frac{\pi x}{L} \right) \sin \left( \frac{\pi y}{L} \right) \right]^{100},
\]
and observe what happens to the peak as it is advected around by the divergence-free flow. Comment on your observations.