Computational Fluid Dynamics, Fall 2014
Makeup Homework: Advection-Diffusion Equations

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Consider numerically solving the advection-diffusion equation

$$u_t + au_x = \left(d(x)u_x\right)_x,$$

on the periodic domain $0 \leq x < 1$, for $a = 1$ and initial condition

$$u(x, 0) = \left[\sin\left(\pi x\right)\right]^{100},$$

on a uniform grid.

1 Spatial Discretization

Here we focus on spatial discretization, and do not discretize time but rather study the consistency and stability of the semi-discrete approximation

$$\boldsymbol{w}'\left(t\right) = \boldsymbol{A}\boldsymbol{w}\left(t\right) + \boldsymbol{g}(t), \tag{1}$$

where $\boldsymbol{A}$ depends on the choice of the spatial discretization. Use a discrete Fourier Transform (DFT) to diagonalize the system of ODEs (1) and then solve it numerically using the FFT algorithm.

1. Choose a good spatial discretization and write it down. Explain what you chose, what its advantages and disadvantages are, and what its order is.
2. Adding a small amount of constant diffusion, $d(x) = 0.001$, compare the numerical solution $\boldsymbol{w}(t = 1)$ to the exact solution $u(x, 1)$ (explain how you computed the "exact" solution to roundoff tolerance) for several resolutions and comment on your observations.
3. Compute the *relative* global error $\|\epsilon(t)\| = \|\boldsymbol{w}(t) - \boldsymbol{u}_h(t)\| / \|\boldsymbol{u}_h(t)\|$ at time $t = 1$ for different grid resolutions, and estimate the spatial order of accuracy empirically.

2 Spatio-Temporal Discretization

Repeat part 1 but now use the Lax-Wendroff or Fromm's (recommended, it may be useful to recall the derivation of Fromm's method from class based on extrapolating face-centered state at the midpoint in time) method for the advection, and make the diffusion coefficient non-constant,

$$d(x) = 0.05\left(2 + \cos(2\pi x)\right).$$

In this case, it is a bit harder to compute an exact solution (please do not try, this is *not* the point of this homework!).

1. Write down a spatio-temporal discretization. Explain how you handled diffusion in your spatio-temporal discretization and what you expect the order of accuracy of the method to be (do your best to make it second-order, of course).
2. Validate your code in some way (e.g., by solving problem 1 using the new method, or, if you know how, using a manufactured analytical solution).
3. Refine the resolution in *both space and time* (i.e., in space-time, not space or time separately) to empirically estimate the spatio-temporal order of convergence.
4. (Optional) Investigate the stability of your scheme – is it limited in stability by both advection and diffusion or only advection?
   Note: You may find that Fromm's method behaves differently from Lax-Wendroff.