Consider numerically solving the advection-diffusion equation
\[ u_t + au_x = (d(x)u_x)_x, \]
on the periodic domain \( 0 \leq x < 1 \), for \( a = 1 \) and initial condition
\[ u(x, 0) = [\sin (\pi x)]^{100}, \]
on a uniform grid.

1 Spatial Discretization
Here we focus on spatial discretization, and do not discretize time but rather study the consistency and stability of the semi-discrete approximation
\[ w'(t) = Aw(t) + g(t), \tag{1} \]
where \( A \) depends on the choice of the spatial discretization. Use a discrete Fourier Transform (DFT) to diagonalize the system of ODEs (1) and then solve it numerically using the FFT algorithm.

1. Choose a good spatial discretization and write it down. Explain what you chose, what its advantages and disadvantages are, and what its order is.
2. Adding a small amount of constant diffusion, \( d(x) = 0.001 \), compare the numerical solution \( w(t = 1) \) to the exact solution \( u(x, 1) \) (explain how you computed the “exact” solution to roundoff tolerance) for several resolutions and comment on your observations.
3. Compute the relative global error \( \| \epsilon(t) \| = \| w(t) - u_h(t) \| / \| u_h(t) \| \) at time \( t = 1 \) for different grid resolutions, and estimate the spatial order of accuracy empirically.

2 Spatio-Temporal Discretization
Repeat part 1 but now use the Lax-Wendroff or Fromm’s (recommended, it may be useful to recall the derivation of Fromm’s method from class based on extrapolating face-centered state at the midpoint in time) method for the advection, and make the diffusion coefficient non-constant,
\[ d(x) = 0.05 (2 + \cos(2\pi x)). \]
In this case, it is a bit harder to compute an exact solution (please do not try, this is not the point of this homework!).

1. Write down a spatio-temporal discretization. Explain how you handled diffusion in your spatio-temporal discretization and what you expect the order of accuracy of the method to be (do your best to make it second-order, of course).
2. Validate your code in some way (e.g., by solving problem 1 using the new method, or, if you know how, using a manufactured analytical solution).
3. Refine the resolution in both space and time (i.e., in space-time, not space or time separately) to empirically estimate the spatio-temporal order of convergence.
4. (Optional) Investigate the stability of your scheme – is it limited in stability by both advection and diffusion or only advection?
   Note: You may find that Fromm’s method behaves differently from Lax-Wendroff.